1) Problema di Cauchy: $y' = \frac{y^{-3}e^x}{4}$; $y(\ln(24)) = -2$

E' una E.D. a variabili separabili, " $y' = f(x)\cdot g(y)$ " con:

$f(x) = e^x$, $x \in I =]-\infty,+\infty[$; $g(y) = \frac{y^{-3}}{4}$, $y \in]-\infty,0[$ (perché $y_0 = -2 < 0$)

$\frac{dy}{dx} = \frac{y^{-3}e^x}{4}$; $\int 4y^3\,dy = \int e^x\,dx$; $y^4 = e^x + C$;

$\left.\begin{array}{l} x=\ln(24)\\ y=-2\end{array}\right\} \to 16 = 24 + C$; $C = -8$, quindi $y^4 = e^x - 8$, e allora

tenendo presente che y deve avere valori <0, $\boxed{y(x) = -\sqrt[4]{e^x-8}}$

Dominio I_0 della soluzione: $e^x - 8 > 0$, $e^x > 8$. $x > \ln 8$: $I_0 =]\ln 8, +\infty[$

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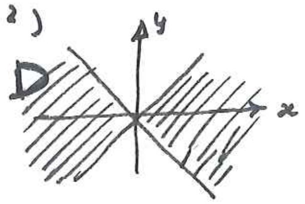
2) DOMINIO, PUNTI CRITICI PER $f(x,y) = 4x + 2y + \ln(x^2 - y^2)$



Dominio: $D = \{(x,y)\in\mathbb{R}^2;\ x^2 - y^2 > 0\}$

$\begin{cases} f'_x = 4 + \frac{2x}{x^2-y^2} = 0 & (1) \to 4(x^2-y^2)+2x=0 \to x^2-y^2 = -\frac12 x \\ f'_y = 2 - \frac{2y}{x^2-y^2} = 0 & (2) \to 2 - 2y\cdot(-\frac{2}{x}) = 0 \to \frac{y}{x} = -\frac12 \to y = -\frac12 x \ (*)\end{cases}$

allora (1) $\to x^2 - \frac14 x^2 = -\frac12 x$: $x(\frac34 x + \frac12) = 0$, $x=0 \vee x = -\frac23$

Se $x=0$, $* \Rightarrow y=0$; $(0,0)\notin D$. Se $x = -\frac23$, $* \Rightarrow y = \frac13$;

$(-\frac23, \frac13) \in D$, è l'unico punto critico per f. Matrice Hessiana:

$f''_{xx} = \frac{2x^2-2y^2-4x^2}{(x^2-y^2)^2} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}$; $f''_{xy} = 2x\cdot(-1)(x^2-y^2)^{-2}\cdot(-2y) = \frac{4xy}{(x^2+y^2)^2}$

$f''_{yx} = -2y(-1)(x^2-y^2)^{-2}\cdot 2x = \frac{-4xy}{(x^2-y^2)^2}$; $f''_{yy} = \frac{-2x^2+2y^2-4y^2}{(x^2-y^2)^2} = \frac{-2x^2-2y^2}{(x^2-y^2)^2}$

$H(x,y) = \frac{1}{(x^2-y^2)^2}\begin{pmatrix} -2x^2-2y^2 & 4xy \\ 4xy & -2x^2-2y^2\end{pmatrix}$; $H(-\frac23,\frac13) = \begin{pmatrix}-10 & -8\\ -8 & -10\end{pmatrix}$

det H = 36 > 0, $f''_{xx} = -10 < 0$ $\Rightarrow$ $(-\frac23,\frac13)$ punto di MASSIMO RELATIVO per f

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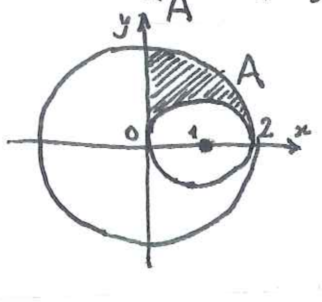
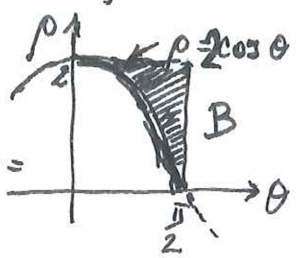
3) * $\iint_A (x^2y+y^3)\,dx\,dy$; $A = \{(x,y)\in\mathbb{R}^2;\ x^2+y^2\le 4;\ x^2+y^2-2x\ge 0,\ x\ge0,\ y\ge0\}$



Usiamo le coordinate polari $\begin{cases}x=\rho\cos\theta\\ y=\rho\,\mathrm{sen}\theta\end{cases}$.

"$x^2+y^2\le4$" dice che $\rho\le2$; "$x^2+y^2-2x\ge0$" dice che $\rho^2-2\rho\cos\theta\ge0$, quindi $\rho\ge2\cos\theta$; $x\ge0, y\ge0$ dicono che $0\le\theta\le\frac\pi2$. Quindi $B = \{(\rho,\theta);\ 0\le\theta\le\frac\pi2;\ 2\cos\theta\le\rho\le2\}$

$\iint_B \rho^2\cdot\rho\,\mathrm{sen}\theta\cdot\rho\,d\rho\,d\theta = \int_0^{\pi/2}\left(\int_{2\cos\theta}^{2} \rho^4\,\mathrm{sen}\theta\,d\rho\right)d\theta$



$= \int_0^{\pi/2}\left[\frac15\rho^5\,\mathrm{sen}\theta\right]_{\rho=2\cos\theta}^{\rho=2}d\theta = \int_0^{\pi/2}\left(\frac{32}{5}\mathrm{sen}\theta - \frac{32}{5}\cos^5\theta\,\mathrm{sen}\theta\right)d\theta =$

$= \left[-\frac{32}{5}\cos\theta + \frac{32}{30}\cos^6\theta\right]_{\theta=0}^{\theta=\pi/2} = \frac{32}{5} - \frac{16}{15} = \frac{96-16}{15} = \frac{80}{15} = \frac{16}{3}$

VEDI OLTRE, RISOLUZIONE IN COORDINATE CARTESIANE

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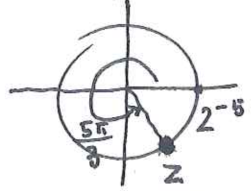
4) MODULO, ARGOMENTO, Re z, Im z per $z = (1-i\sqrt3)^{-5}$

Conviene scrivere $1-i\sqrt3$ in forma trigonometrica: $|1-i\sqrt3| = \sqrt4 = 2$;

$\arg(1-i\sqrt3) = \arctan(-\sqrt3) = -\frac\pi3$; quindi $1-i\sqrt3 = 2e^{-\frac\pi3 i}$; allora

$z = (1-i\sqrt3)^{-5} = 2^{-5}\cdot e^{\frac53\pi i} = 2^{-5}\left(\cos\left(\frac{5\pi}{3}\right) + i\,\mathrm{sen}\left(\frac{5\pi}{3}\right)\right)$.



Si nota allora che: $|z| = 2^{-5}$; $\arg z = \frac{5\pi}{3}$ e

$\mathrm{Re}\,z = 2^{-5}\cos\left(\frac{5\pi}{3}\right) = 2^{-5}\cdot\frac12 = 2^{-6} = \frac{1}{64}$; $\mathrm{Im}\,z = 2^{-5}\,\mathrm{sen}\,\frac{5\pi}{3} = -\frac{\sqrt3}{64}$

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5) $A = \begin{pmatrix}1&-1&a\\0&2&b\\-1&-1&c\end{pmatrix}$; DETERMINARE a, b, c IN MODO CHE $\underline{v} = \begin{pmatrix}1\\0\\1\end{pmatrix}$ SIA AUTOVETTORE PER A CON AUTOVALORE 1. CON TALI a, b, c DETERMINARE AUTOVALORI E AUTOVETTORI DI A, E DIRE SE A E' DIAGONALIZZABILE SU $\mathbb{R}$.

Deve essere $(A-I)\cdot\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, cioè $\begin{cases}a=0\\b=0\\-1+c-1=0\end{cases} \to \begin{cases}a=0\\b=0\\c=2\end{cases}$; quindi:

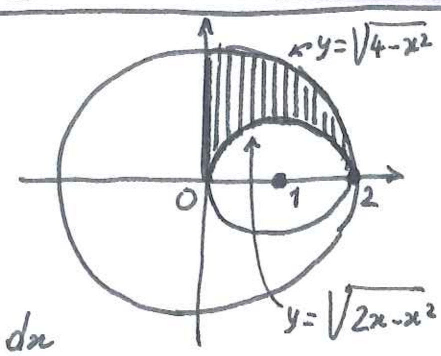
$A = \begin{pmatrix}1&-1&0\\0&2&0\\-1&-1&2\end{pmatrix}$; $\det\begin{pmatrix}1-\lambda&-1&0\\0&2-\lambda&0\\-1&-1&2-\lambda\end{pmatrix} = (2-\lambda)^2(1-\lambda)$; autovalori $\lambda=2$ DOPPIO, $\lambda=1$ SEMPLICE

Autospazio relativo a $\lambda=1$, necessariamente di dimensione 1, è noto: $S_1 = \mathrm{Span}\left(\begin{pmatrix}1\\0\\1\end{pmatrix}\right)$. L'autospazio relativo a $\lambda=2$ si trova risolvendo:

$(A-2I)\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ cioè $\begin{cases}-x-y=0\\0=0\\-x-y=0\end{cases} \iff y=-x$; gli autovettori sono $(x,-x,z)$; con $x, z\in\mathbb{R}$ arbitrari; cioè:

$S_2 = \mathrm{Span}\left(\begin{pmatrix}1\\-1\\0\end{pmatrix};\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$ Poiché $\dim S_2 = 2$, A è diagonalizzabile su $\mathbb{R}$, ed è simile alla matrice $D = \begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$.

RISOLUZIONE DI 3) CON COORDINATE CARTESIANE



$\iint_A(x^2y+y^3)\,dx\,dy = \int_0^2\left(\int_{\sqrt{2x-x^2}}^{\sqrt{4-x^2}}(x^2y+y^3)\,dy\right)dx =$

$= \int_0^2\left[\frac12x^2y^2+\frac14y^4\right]_{y=\sqrt{2x-x^2}}^{y=\sqrt{4-x^2}}dx =$

$= \int_0^2\left[\frac12x^2(4-\cancel{x^2}-2x+\cancel{x^2}) + \frac14((4-x^2)^2-(2x-x^2)^2)\right]dx$

$= \int_0^2\left(2x^2-x^3+\frac14(16+\cancel{x^4}-8x^2-4x^2-\cancel{x^4}+4x^3)\right)dx =$

$= \int_0^2(\cancel{2x^2}-\cancel{x^3}+4-\cancel{2x^2}-x^2+\cancel{x^3})\,dx = \int_0^2(4-x^2)\,dx =$

$= \left[4x-\frac13x^3\right]_{x=0}^{x=2} = 8-\frac83 = \frac{16}{3}$