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COMPUTING THE NUMBER OF APPARENT DOUBLE POINTS OF A SURFACE

Dedicated to Alberto Conte on the occasion of his 70th birthday

Abstract. For a smooth surface $S \subset \mathbb{P}^5_K$ there are well known classical formulas giving the number $\rho(S)$ of secants of S passing through a generic point of $\mathbb{P}^5$. In this paper, for possibly singular surfaces T, a computer assisted computation of $\rho(T)$ from the defining ideal $I(T) \subset K[x_0,\dots,x_5]$ is proposed. It is based on the Stückrad-Vogel self-intersection cycle of T and requires the computation of the normal cone of the ruled join $J(T,T)$ along the diagonal. It is shown that in the case when $T \subset \mathbb{P}^5$ arises as the linear projection with center L of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ (which satisfies some mild assumptions), the computational complexity can be reduced considerably by using the normal cone of $\operatorname{Sec} S$ along $L \cap \operatorname{Sec} S$ instead of the former normal cone. Many examples and the relative code for the computer algebra systems REDUCE, CoCoA, Macaulay2 and Singular are given.

1. Introduction

For a smooth surface $S \subset \mathbb{P}^5_K$, where K is an algebraically closed field of characteristic zero, there are well known classical formulas giving the number $\rho(S)$ of secants of S passing through a generic point of $\mathbb{P}^5$ (see, for example, the double point formula of Severi [16], p. 259 or [22], Example 4.1.3, or the secant formula of Peters-Simonis [26]). This number is called the *secant order* or the *number of apparent double points* of S, since it is the number of double points of the projection of S into $\mathbb{P}^4$ from a generic point of $\mathbb{P}^5$. These formulas are not suited if one wants to compute the number ρ starting from the equations of the surface. In this paper we provide a computational approach based on the coefficients of a certain Hilbert polynomial which comes from the Stückrad-Vogel intersection cycle and can be computed from the equations of the surface.

In [5] (see Theorem 4.3) it was shown that for a singular non-defective surface $T \subset \mathbb{P}^5_K$ the Stückrad-Vogel self-intersection cycle of T can be used to obtain a formula for $\rho(T)$. From a computational point of view, this result permits to compute $\rho(T)$ using computer algebra systems, but it requires the computation of the normal cone of the ruled join $J(T,T)$ along the diagonal, the computational complexity of which can be very high.

In this paper we propose a second method for the computation of $\rho(T)$ when the singular surface $T \subset \mathbb{P}^5$ arises as the projection of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ along a linear subspace L. Under some mild assumption on S, this method reduces the computational complexity of the previous one, since it relies on the computation of the normal cone of $\operatorname{Sec} S$ along $L \cap \operatorname{Sec} S$ (see Theorem 3).

For some special surfaces the computational complexity can be further reduced by applying an observation of A. Verra. He observed that for certain possibly singular surfaces S (which C. Ciliberto and F. Russo called *Verra surfaces*, see Definition 2) the number $\rho(S)$ can be obtained computing the number of apparent double points of a space curve by using its self-intersection cycle (see [4], Proposition 3.7). In the case of such surfaces the computation of ρ can be done in $\mathbb{P}^3$, that is in a ring with only four variables.

We illustrate the methods through a collection of examples, where the computations have been done with REDUCE (see [1]), but they could have been done with other systems as CoCoA, Macaulay2, Singular as well.

In the last section of the paper we give the code of the procedures for the calculations in the different computer algebra systems and we show the efficiency of the new method given by Theorem 3.

2. Computational aspects of the Stückrad-Vogel intersection cycle

Let X, Y be closed (irreducible and reduced) subvarieties of the projective space $\mathbb{P}^N = \mathbb{P}^N_K$, where K is an algebraically closed field of characteristic zero. Stückrad and Vogel [30] (see also [16], Section 2.2) introduced a cycle $v(X,Y)$ called v-cycle, which is the formal sum of (algorithmically produced) subvarieties C of $X \cap Y$ (possibly defined over a pure transcendental field extension of K), each taken with an algorithmically produced positive integer coefficient $j_C = j(X,Y;C)$, the *intersection multiplicity* of X and Y along C. In order to describe the dimension range of the varieties C, denote by XY or $\mathrm{embJ}(X,Y)$ the *embedded join* of X and Y, that is, the closure of the union of all lines $\overline{xy}$, $x \in X$, $y \in Y$. Then the v-cycle can be written as

$$v(X,Y) = \sum_C j(X,Y;C)[C] = \sum_k v_k(X,Y),$$

where k runs from $\max(\dim X + \dim Y - \dim XY, 0)$ to $\dim X \cap Y$, and $v_k(X,Y) (\neq 0)$ is the k-dimensional part of $v(X,Y)$.

The part of the cycle $v_k = v_k(X,Y)$ which is defined over the base field K, the so-called *K-rational part*, will be denoted by $\mathrm{rat}(v_k)$ and the remaining part, the so-called *irrational* or *movable part*, will be denoted by $\mathrm{mov}(v_k)$, that is,

$$v_k = \mathrm{rat}(v_k) + \mathrm{mov}(v_k).$$

To prove a theorem of Bezout, Stückrad and Vogel had to take into account also the so-called empty subvariety $\emptyset \subseteq X \cap Y$ (which by definition has dimension -1 and degree 1) and its intersection number $j(X,Y,\emptyset)$.

We want to describe $j(X,Y,\emptyset)$ following van Gastel [19]. To this end, let $A_x := K[x_0,\ldots,x_N]$, $A_y := K[y_0,\ldots,y_N]$, $A := A_x \otimes_K A_y$ and denote by $I(X) \subseteq A_x$, $I(Y) \subseteq A_y$ the largest homogeneous ideals defining X and Y, respectively. Then $R := A/(I(X)A + I(Y)A)$ is the homogeneous coordinate ring of the *ruled join* $J = J(X,Y) \subseteq \mathbb{P}^{2N+1}_K = \mathrm{Proj}(A)$. The "diagonal" subspace Δ of $\mathbb{P}^{2N+1}_K$ is defined by the ideal $(x_0 -$

$y_0,\dots,x_N-y_N)A =: I(\Delta)$. Denoting by $\deg(J(X,Y)/XY)$ the mapping degree of the linear projection $J(X,Y) \dashrightarrow XY$ with center $\Delta\cap J(X,Y)$, van Gastel proved that

$$j(X,Y;\emptyset) = \deg(J/XY)\deg XY.$$

Then, using the above notation, the *refined Bezout theorem* (see [16], Theorem 2.2.5) can be formulated as follows.

THEOREM 1 (Stückrad-Vogel [30], van Gastel [19]). *Let X,Y be closed (irreducible and reduced) subvarieties of $\mathbb{P}^N_K$. Then*

$$\begin{aligned}\deg X\deg Y &= \deg(J/XY)\deg XY + \sum_C j(X,Y;C)\deg C\\ &= \deg(J/XY)\deg XY + \deg v(X,Y)\\ &= \deg(J/XY)\deg XY + \sum_k \deg v_k(X,Y).\end{aligned}$$

The degrees of the v_k's and $\deg(J/XY)\deg XY$ can be calculated by the Hilbert coefficients of the bigraded ideal which defines the normal cone of $J(X,Y)$ along $J(X,Y)\cap\Delta$. More precisely, let

$$A = K[x_0,\dots,x_N,y_0,\dots,y_N], \quad R = A/I(J(X,Y)),$$
$$I = I(\Delta) = (x_0-y_0,\dots,x_N-y_N)A,$$

$t_0,\dots,t_N$ indeterminates, and

$$\varphi\colon A[t_0,\dots,t_N] \to G_I(R) := \bigoplus_{k\in\mathbb{N}} I^k/I^{k+1}$$

be the natural surjection which is induced by the natural homomorphism $A\to R/I$ and substituting t_i by the class of x_i-y_i in I/I^2, $i=0,\dots,N$. Define a bigrading by setting $\deg(x_i)=\deg(y_i)=(1,0)$ and $\deg(t_i)=(0,1)$. Then $\ker\varphi$ is a bigraded ideal in the bigraded ring $A[t_0,\dots,t_N]$, and

(1) $$\mathcal{R} := \mathcal{R}(X,Y) := G_I(R) \cong A[t_0,\dots,t_N]/\ker\varphi$$

is bigraded, $\mathcal{R} = \bigoplus_{j,k\in\mathbb{N}}\mathcal{R}_{j,k}$. If we set $d := \text{Krull-dim}\, R = \text{Krull-dim}\,\mathcal{R}$ and write the Hilbert polynomial of $\sum_{v=0}^k\sum_{u=0}^j \dim_K(\mathcal{R}_{u,v})$ in the form

$$p(j,k) = \sum_{l=0}^d c_l\binom{j+l}{l}\binom{k+d-l}{d-l} + \text{ lower degree terms},$$

then the non-negative integers $c_l =: c_l(\mathcal{R}) =: c_l(X,Y)$ are the *generalized Samuel multiplicities* of I in the sense of [3].

PROPOSITION 1 ([3], Theorem 4.1 and Proposition 1.2). *With the previous notation,*

$$c_0(\mathcal{R}) = c_0(X,Y) = j(X,Y;\emptyset) = \deg(J/XY)\deg XY$$

and for $k = 1, \ldots, d$*,*

$$c_k(\mathcal{R}) = c_k(X,Y) = \sum_{\mathcal{P} \in \operatorname{minAss} \mathcal{R}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_k(\mathcal{R}/\mathcal{P}) = \deg v_k(X,Y).$$

According to (1) the minimal prime ideals $\mathcal{P}$ of $\mathcal{R}$ contract to prime ideals $\mathcal{P} \cap A$ which contain the ideals $I(\Delta)$ and $I(J(X,Y))$, hence the contraction ideals $\mathcal{P} \cap K[x_0, \ldots x_N]$ (which need not all be distinct) define subvarieties of $X \cap Y \subset \mathbb{P}^N$, the so called *distinguished varieties* of the intersection of X and Y in the sense of Fulton [17]. These subvarieties are the support of the K-rational part of $v(X,Y)$. The lengths of the $\mathcal{R}_{\mathcal{P}}$'s are the geometric multiplicities of the irreducible components of the normal cone of $J(X,Y)$ along $J(X,Y) \cap \Delta$.

DEFINITION 1 (Intersection vector). *With the above notation, set* $\delta := \dim(X \cap Y) + 1$*. Then the* intersection vector $\mathbf{c}(X,Y)$ *of* X *and* Y *is defined to be the vector of non-negative integers*

$$\mathbf{c}(X,Y) = (c_0(X,Y), \ldots, c_\delta(X,Y)) = (c_0(\mathcal{R}), \ldots, c_\delta(\mathcal{R})) =: \mathbf{c}(\mathcal{R}),$$

and by the refined Bezout theorem

$$\deg X \deg Y = c_0(X,Y) + \cdots + c_\delta(X,Y).$$

By Proposition 1 we have

$$\mathbf{c}(X,Y) = \sum_{\mathcal{P} \in \operatorname{minAss} \mathcal{R}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot \mathbf{c}(\mathcal{R}/\mathcal{P}). \tag{2}$$

In particular, the self-intersection vector *of an* n*-dimensional variety* X *is defined to be*

$$\mathbf{c}(X) = \mathbf{c}(X,X) = (c_0(X,X)), \ldots, c_{n+1}(X,X)),$$

and it holds

$$(\deg X)^2 = c_0(X,X) + \cdots + c_\delta(X,X).$$

The integers $c_k(X,Y) = c_k(\mathcal{R})$ can be computed by using various computer algebra systems (e.g. REDUCE, using the package SEGRE [1]), in which the calculation of the Hilbert series of a multigraded ring has been implemented, see Section 5.

For the computation of the number of apparent double points of a variety X, the coefficient

$$c_0(X,X) = j(X,X;\emptyset) = \deg(J(X,X)/\operatorname{embJ}(X,X)) \cdot \deg(\operatorname{embJ}(X,X))$$

is particularly important. Note that $\operatorname{embJ}(X,X)$ is the *secant variety*, which we denote by $\operatorname{Sec} X$. It is well-known that $\operatorname{Sec} X$ has the *expected dimension* $2\dim X + 1$ (and is said to be *nondeficient*) if and only if for generic points $x \in X$ and $y \in X$ one has $T_{X,x} \cap T_{X,y} = \emptyset$ (see, for example, [16], Cor. 4.3.3). The non-negative integer $2\dim X + 1 - \dim \operatorname{Sec} X$ is called the *deficiency* of $\operatorname{Sec} X$. Concerning $\deg(J(X,X)/\operatorname{Sec} X)$, there is the following result, which is essentially [16], Proposition 8.2.12, see also [24], Proposition 6.3.5.

PROPOSITION 2 ([5], Proposition 2.7). *Let $X \subset \mathbb{P}^N$ be a non-degenerate irreducible subvariety such that $2\dim X + 1 < N$. Suppose that one of the following two conditions is satisfied:*

1. *X is a curve;*

2. *X is reduced and the generic tangent hyperplane to $\operatorname{Sec} X$ is tangent to X at only finitely many points (that is, X is not 1-weakly defective in the sense of [9], 2.1).*

Then $\deg(J(X,X)/\operatorname{Sec} X) = 2$ *and, in particular,* $\dim \operatorname{Sec} X = 2\dim X + 1$.

REMARK 1. If $x, y \in X$, then over the secant line xy of X there are two lines $J(x,y)$, $J(y,x)$ of $J(X,X)$, so that the rational map

$$\pi\colon J(X,X) \dashrightarrow \operatorname{Sec} X,$$

has even degree, that is,

$$\deg(J(X,X)/\operatorname{Sec} X) = 2\rho \geq 0.$$

Here ρ is the *number of secants to X passing through a general point of* $\operatorname{Sec} X$, if $\dim \operatorname{Sec} X = 2\dim X + 1$, and $\rho = 0$ otherwise. In [9] one can find a complete classification of surfaces $X \subset \mathbb{P}^N$, $N \geq 6$, for which $\rho > 1$.

3. Computing ρ by the Stückrad-Vogel intersection cycle

Let $S \subset \mathbb{P}^N$ ($N \geq 5$) be a non-degenerate surface of degree d with singular locus $\operatorname{Sing} S$. For any point $P \in \operatorname{Sm}(S)$ we denote by $T_{S,P}$ the embedded projective tangent plane to S at P. We denote by $\operatorname{Tan} S$ the tangent variety of S, that is the closure of the union of all embedded projective tangent planes to S at regular points.

It is known (see, for example, [5]) that the Stückrad-Vogel self-intersection cycle of S is

$$v(S,S) = [S] + \sum_Z j_Z[Z] + P_1(S) + \sum_P j_P[P] + \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} + \operatorname{mov} v_0(S)|_{\operatorname{Sing} S}, \tag{3}$$

where Z runs through the one-dimensional irreducible components of $\operatorname{Sing} S$, $P_1(S)$ denotes the first polar locus of S, P runs through the singular points of S of embedding dimension greater or equal to 4, and $\operatorname{mov} v_0(S)|_{\operatorname{Sm} S}$ is the ramification locus of the linear projection $\pi_\Lambda\colon S \to \mathbb{P}^3$ with center a generic $(N-4)$-dimensional linear subspace $\Lambda \subset \mathbb{P}^N$ (see [15], Theorem 4.6).

Moreover,

$$\deg v_1(S) = \sum_Z j_Z \deg Z + \deg P_1(S) = c_2(S,S),$$

$$\deg v_0(S) = \sum_P j_P + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} = c_1(S,S),$$

and, if S is non defective,

$$\deg \operatorname{mov} v_0(S)|_{\operatorname{Sm} S} = \deg(\operatorname{Tan} S)$$

(see [5], Lemma 4.1).

In [5], Theorem 4.3, the following has been proved.

THEOREM 2. *Let $S \subset \mathbb{P}^N$ ($N \geq 5$) be an irreducible and reduced possibly singular non defective surface of degree d. With the preceding notation the following formula holds:*

$$\begin{aligned} 2\rho \cdot \deg \operatorname{Sec} S &= (\deg S)^2 - \deg v(S,S) \\ &= (\deg S)^2 - c_3(S,S) - c_2(S,S) - c_1(S,S) \\ &= d^2 - d - \sum_Z j_Z \deg Z - \deg P_1(S) \\ &\quad - \sum_P j_P - \deg \operatorname{Tan} S - \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} \\ &= c_0(S,S), \end{aligned}$$

where Z runs through the one-dimensional irreducible components of $\operatorname{Sing} S$ *and P runs through the singular points of S of embedding dimension greater or equal to* 4.

For $N = 5$ one has

$$\rho(S) = \frac{1}{2} c_0(S,S).$$

REMARK 2. The self-intersection vector $v(S,S)$ of a surface $S \subset \mathbb{P}^N, N \geq 5$, encodes geometric information on S. If $S \subset \mathbb{P}^N$ ($N \geq 5$) is smooth and non defective, then the self-intersection vector of S is

$$\begin{aligned} c(S,S) &= (2\rho \deg(\operatorname{Sec} S), \deg(\operatorname{Tan} S), \deg(P_1(S)), \deg(S)) = \\ &= (2\rho \deg(\operatorname{Sec} S), 0, 0, 0) + (0, \deg(\operatorname{Tan} S), \deg(P_1(S)), \deg(S)), \end{aligned}$$

where the last line is the decomposition according to Proposition 1. In fact, for S smooth and non defective the ring of coordinates of the normal cone $\mathcal{R}$ has two minimal primes $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{P}_1 \cap K[x_0, \ldots, x_N] = (x_0, \ldots, x_N)$ and $\mathcal{P}_2 \cap K[x_0, \ldots, x_N] = I(S)$.

If the surface $S \subset \mathbb{P}^N, N \geq 5$, has isolated singular points $P_1, \ldots, P_r$ of embedding dimension greater or equal to 4 and singular curves $Z_1, \ldots, Z_s$, then the normal cone of $J(S,S)$ along $J(S,S) \cap \Delta$ has at least one component for each point P_i, $1 \leq i \leq r$ and for each curve Z_j, $1 \leq j \leq s$. The self-intersection vector of S decomposes in the following way

$$\begin{aligned}
c(S,S) = \Bigg(& 2\rho \deg(\operatorname{Sec} S),\ \deg(\operatorname{Tan} S) + \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S} + \sum_P j_P, \\
& \qquad \deg(P_1(S)) + \sum_Z j_Z \deg(Z),\ \deg(S) \Bigg) \\
= {} & (2\rho \deg(\operatorname{Sec} S),\ 0,\ 0,\ 0) + \\
& + \sum_{P \in \{P_i\}} \Bigg(0, \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(P)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_1(\mathcal{R}/\mathcal{P}), 0, 0 \Bigg) + \\
& + \sum_{Z \in \{Z_j\}} \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot (0,\ c_1(\mathcal{R}/\mathcal{P}),\ c_2(\mathcal{R}/\mathcal{P}),\ 0) + \\
& + (0,\ \deg(\operatorname{Tan} S),\ \deg(P_1(S)),\ \deg(S)),
\end{aligned}$$

where we recall that $A_x := K[x_0,\dots,x_N]$, and we remark that for the movable points Q on the curves Z_j the coefficients $\operatorname{length}(\mathcal{R}_{\mathcal{P}})$ are equal to the intersection number j_Q and

$$\sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_2(\mathcal{R}/\mathcal{P}) = j_Z \deg(Z)$$

(see [6], Main Theorem). We also observe that

$$\sum_{Z \in \{Z_j\}} \sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} \operatorname{length}(\mathcal{R}_{\mathcal{P}}) \cdot c_1(\mathcal{R}/\mathcal{P}) = \deg \operatorname{mov} v_0(S)|_{\operatorname{Sing} S},$$

and $\sum_{\{\mathcal{P} | \mathcal{P} \cap A_x = I(Z)\}} c_1(\mathcal{R}/\mathcal{P})$ is the number of the movable points on Z.

In order to compute the intersection number j_P of an isolated singular point of S we must compute the generalized Samuel multiplicities of the diagonal ideal in the localization of the ring R localized at the prime ideal $I(P)R + I(\Delta)R$ (see [3]). In this case we obtain three coefficients: $c_0 = j_P, c_1, c_2$, where $c_0 = 0$ if and only if the embedding dimension of S at P is smaller or equal to 3 and c_2 is the multiplicity of S at P.

PROPOSITION 3. *(C.Ciliberto) A surface $S \subset \mathbb{P}^5$ with one apparent double point ($\rho(S) = 1$) cannot have isolated singular points of embedding dimension greater or equal to* 4.

Proof. In fact, assume that $P \in S$ is a point of embedding dimension greater or equal to 4 and Π is the Zariski tangent space to S at P. If r is a generic secant line of S and $\alpha = \langle r, P \rangle \cong \mathbb{P}^2$, then $\alpha \cap \Pi$ contains at least a line ℓ through P. The line $\ell \subset \Pi$ is a limit of secants, hence it is a secant, but $r \cap \ell$ is not empty, which contradicts the genericity of r. $\square$

This says that we cannot have surfaces in $\mathbb{P}^5$ with one apparent double point and singular points which contributes to the self-intersection cycle. We can have such examples only if $\rho \geq 2$.

By using Theorem 2, we want to compute the self-intersection vector and the number $\rho(S)$, for singular surfaces in $S \subset \mathbb{P}^5$ which are linear projections of rational normal scrolls. We also want to point out the contribution of the components of the singular locus of S to its self-intersection vector.

In the following with $S(a,b) \subset \mathbb{P}^{a+b+1}$ we denote the rational normal scroll defined by the parametric equations

$$(x_0 : \ldots : x_a : y_0 : \ldots : y_b) = (s^a u : s^{a-1}tu : \ldots : t^a u : s^b v : s^{b-1}tv : \ldots : t^b v).$$

We recall that $S(a,b)$ is a smooth surface of degree $a+b$, whose defining equations are given by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & \ldots & x_{a-1} & y_0 & \ldots & y_{b-1} \\ x_1 & x_2 & \ldots & x_a & y_1 & \ldots & y_b \end{pmatrix}$$

(for more details see, for example, [21]). The defining ideals of $S(a,b)$ can be conveniently computed by various computer algebra systems, see Section 5.

By [5], the self-intersection vector of $S(a,b)$ is

$$\begin{aligned} c(S,S) &= (2\deg(\operatorname{Sec} S),\ \deg(\operatorname{Tan} S),\ \deg(P_1(S)),\ \deg(S)) \\ &= ((a+b-2)(a+b-3),\ 2(a+b-2),\ 2(a+b-1),\ a+b). \end{aligned} \tag{4}$$

Now we are ready to present some examples, where the self-intersection vectors have been computed following procedures and codes as explained in Section 5.

EXAMPLE 1. *Surfaces in $\mathbb{P}^5$ with one or two isolated singular points which do not contribute to the self intersection cycle and one apparent double point*
Let us now consider the smooth del Pezzo surface of $S \subset \mathbb{P}^6$ given by the parametric equations

$$x_0 = uvw, x_1 = v^2w, x_2 = vw^2, x_3 = uw^2, x_4 = u^2w, x_5 = u^2v, x_6 = uv^2$$

(see [28], p. 155). The secant variety of S is the hypersurface of $\mathbb{P}^6$ defined by the equation

$$x_0^3 - x_0x_1x_4 - x_0x_2x_5 - x_0x_3x_6 + x_1x_3x_5 + x_2x_4x_6 = 0,$$

whose singular locus is S.

We project S to $\mathbb{P}^5$ from a point $P \in S$. If

$$\begin{aligned} P \in \{&[0:1:0:0:0:0:0], [0:0:1:0:0:0:0], [0:0:0:1:0:0:0], \\ &[0:0:0:0:1:0:0], [0:0:0:0:0:1:0], [0:0:0:0:0:0:1]\} \end{aligned}$$

the image of S under the projection is a surface $T \subset \mathbb{P}^5$ of degree 5 with two singular points which do not contribute to the self-intersection cycle of the surface. The decomposition of the self-intersection vector of the surface T is

$$c(T,T) = (2,8,10,5) = 2(1,0,0,0) + (0,8,10,5),$$

in particular $\rho = 1$.

If we project the surface S from the point $[0:1:1:0:0:0:0]$, we obtain a surface $T' \subset \mathbb{P}^5$ of degree 5 with one singular point which does not contribute to the self-intersection cycle of the surface. The intersection numbers of the surface T' are again

$$c(T',T') = (2,8,10,5) = 2(1,0,0,0) + (0,8,10,5),$$

in particular $\rho = 1$.

EXAMPLE 2. *Surfaces in $\mathbb{P}^5$ with an isolated singular point which contributes to the self intersection cycle and two apparent double points*
Let $S = S(3,2) \subset \mathbb{P}^6$, whose self intersection vector is $c(S(3,2)) = (6,6,8,5))$ and

$$\operatorname{Sing}\operatorname{Sec} S(3,2) = \operatorname{Sing}\operatorname{Tan} S(3,2) = S(3,2) \cap \{[x_0 : \ldots : x_6] \in \mathbb{P}^6 \mid x_0 = \cdots = x_3 = 0\}.$$

Let T_1 and T_2 be the surfaces of $\mathbb{P}^5$ obtained by projecting S from the points $P_1 = [1:0:0:1:0:0:0] \in \operatorname{Sec} S(3,2) \setminus \operatorname{Tan} S(3,2)$ and $P_2 = [0:1:0:0:0:0:0] \in \operatorname{Tan} S(3,2) \setminus \operatorname{Sing}\operatorname{Tan} S(3,2)$ respectively.

The surfaces T_1 and T_2 have one double point which contributes to the intersection cycle with intersection multiplicity $j_1 = 2$ and $j_2 = 3$ respectively. The decompositions of the intersection vectors are:

$$c(T_1,T_1) = (4,8,8,5) = (0,6,8,5) + 2(0,1,0,0) + (4,0,0,0),$$

$$c(T_2,T_2) = (4,8,8,5) = (0,5,8,5) + 3(0,1,0,0) + 4(1,0,0,0),$$

see formula (2) and Section 5.

EXAMPLE 3. *Surface in $\mathbb{P}^5$ with two isolated singular points which contribute to the self intersection cycle and four apparent double points*
Let us consider the rational normal scroll $S(3,3) \subset \mathbb{P}^7$ and we project it from the line s passing through the points $P = [0:1:\ldots:0]$ and $Q = [0:\ldots:0:1:0]$ (which are smooth points of $\operatorname{Sec} S(3,3)$) on the linear space $\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_6 = 0\} \cong \mathbb{P}^5$.

One obtains a surface $T \subset \mathbb{P}^5$ with two isolated singular points $R_1 = [0:0:0:0:0:1]$ and $R_2 = [1:0:0:0:0:0]$, which are double points. A computer calculation as in [1, file segre4.txt] gives the self-intersection numbers of T

$$j(T,T;R_1) = j(T,T;R_2) = 3$$

and the intersection vector

$$c(T,T) = (8,12,10,6) = 8(1,0,0,0) + 3(0,1,0,0) + 3(0,1,0,0) + (0,6,10,6),$$

whereas the intersection vector of $S(3,3)$ is $c(S(3,3),S(3,3)) = (12,8,10,6)$, see (4).

We recall the definition of Verra surfaces from [11], Section 3, in a slightly modified version.

DEFINITION 2 (Verra surfaces). *Let $Y \subset \mathbb{P}^5$ be a degenerate curve, which spans a linear space V of dimension 3. Take a line $W \subset \mathbb{P}^5$ such that $V \cap W = \emptyset$. Let $C_W(Y)$ be the cone over Y with vertex W. Let $X \subset C_W(Y)$ be an irreducible, non-degenerate, not secant defective surface, which intersects the general ruling $\Pi \cong \mathbb{P}^2$ of $C_W(Y)$ along a line L. This implies that:*

(A1) the projection $p\colon \mathbb{P}^5 \dashrightarrow V$ with center W restricts to X to a dominant map $p|_X : X \dashrightarrow Y$;

(A2) if $L_i, 1 \leq i \leq 2$, are the closures of two general fibers of $p|_X$, then $L_1 \cap L_2 = \emptyset$.

Indeed, (A1) is clear, and (A2) follows, via Terracini's Lemma, from the fact that X is not secant defective. The variety X is called a Verra surface *constructed from the curve Y.*

We point out that, differently from [11], in our definition Y is not required to be a curve with one apparent double point.

PROPOSITION 4 (A. Verra). *With the previous notation we have*

$$\rho(X) = \rho(Y).$$

Proof. Let X be a Verra surface. Let $x \in \mathbb{P}^5$ be a general point, so that $y = p(x)$ is a general point of V. A secant line to X through x is a general secant line to X and projects to a general secant line to Y passing through y. Let $\rho(Y) = m$, then there are m secant lines $\ell_1, \ldots, \ell_m$ through y, and let P_{i1}, P_{i2} $(i = 1, \ldots, m)$ the intersection points of ℓ_i with Y. For each secant line ℓ_i of Y through y there is exactly one secant line of X through x which by p is mapped on ℓ_i. Such a line must be in the 3-dimensional linear space $Z_i = \langle \ell_i \cup W \rangle$, which intersects X along the two lines $L_{ij} \subset \langle P_{ij}, W \rangle, 1 \leq j \leq 2$, the union of which spans Z_i. The assertion follows, since there is only one secant line to $L_{i1} \cup L_{i2}$ passing through $x \in Z_i$.

□

The following two examples regard two families of Verra surfaces with a multiple line the preimage of which in the normalization are a rational normal curve and k lines, respectively.

EXAMPLE 4. *Verra surfaces in $\mathbb{P}^5$ with a multiple line and one apparent double point*

Let us consider a rational normal scroll $S(d-3,3) \subset \mathbb{P}^{d+1}$, with $d \geq 5$, and let us project $S(d-3,3)$ from the linear subspace

$$L = \{[x_0 : \ldots : x_{d-3} : y_0 : \ldots : y_3] \in \mathbb{P}^{d+1} \mid x_0 = x_{d-3} = y_0 = \cdots = y_3 = 0\}$$

of dimension $d-5$ contained in

$$\Pi = \{[x_0 : \ldots : x_{d-3} : y_0 : \ldots : y_3] \in \mathbb{P}^{d+1} \mid y_0 = \cdots = y_3 = 0\} \cong \mathbb{P}^{d-3}$$

and such that it does not intersect the rational normal curve $C(d-3) = S(d-3,3) \cap \Pi$.

The image of $S(d-3,3)$ under the linear projection π_L is a rational surface $T := T(d-3,3) \subset \mathbb{P}^5$ with a multiple line ℓ of multiplicity $d-3$ such that ${\pi_L}^{-1}(\ell) = C(d-3)$.

Clearly the restriction of the projection π_L to

$$\Pi' = \{[x_0 : \ldots : x_{d-3} : y_0 : \ldots : y_3] \in \mathbb{P}^{d+1} \mid x_0 = \cdots = x_{d-3} = 0\} \cong \mathbb{P}^3$$

gives an isomorphism between the rational normal curve $C(3) = S(d-3,3) \cap \Pi'$ and its image in $\mathbb{P}^5$, and the surface T is obtained as in Verra's construction (see Definition 2, [10], Example 5.18 and [11], Section 3).

Since T is obtained from Verra's construction, we know that $\rho(T) = \rho(C(3)) = 1$, hence $\deg(J(T,T)/\operatorname{Sec}(T)) = 2$.

We can observe that for $d = 5$,

$$\operatorname{Sing}\operatorname{Sec} S(2,3) = S(2,3) \cup \{[x_0 : \ldots : x_2 : y_0 : \ldots : y_3] \in \mathbb{P}^6 \mid y_0 = \cdots = y_3 = 0\}$$

and L is a point in $\operatorname{Sing}\operatorname{Sec} S(2,3) \setminus S(2,3)$. The surface $T(2,3)$ has degree 5 and its singular locus is a line of double points whose preimage is exactly the smooth conic

$$S(2,3) \cap \{[x_0 : \ldots : x_2 : y_0 : \ldots : y_3] \in \mathbb{P}^6 \mid y_0 = \cdots = y_3 = 0\}.$$

This example was studied in detail in [5], Section 4, and its self-intersection vector is

$$c(T(2,3), T(2,3)) = (2,6,10,5) = 2(1,0,0,0) + 2(0,2,1,0) + (0,4,8,5).$$

The self-intersection vector of $T(2,3)$ is equal to the the self-intersection vectors of the surfaces T and T' of Example 1, but their decompositions are different.

EXAMPLE 5. *A Verra surface in $\mathbb{P}^5$ with a double line and one apparent double point*

Let us now consider the rational normal scroll $S(1,4) \subset \mathbb{P}^6$ and the 3-dimensional irreducible and reduced variety (remember that embJ denotes the embedded join)

$$X := \operatorname{embJ}(S(1), S(1,4)) = \operatorname{embJ}(S(1), S(4)) = \operatorname{Sing}\operatorname{Sec} S(1,4) = \operatorname{Sing}\operatorname{Tan} S(1,4)$$

of defining ideal

$$(-x_2x_4 + {x_3}^2, -x_2x_5 + x_3x_4, -x_2x_6 + x_3x_5, -x_2x_6 + {x_4}^2, -x_3x_6 + x_4x_5, -x_4x_6 + {x_5}^2).$$

Let $P \in X \setminus S(1,4)$ and let $\pi_P \colon S(1,4) \to \mathbb{P}^5$ be the linear projection from P into $\mathbb{P}^5$. The surface $Z := Z(1,4) := \pi_P(S(1,4))$ has a singular line $\ell = \pi_P(S(1))$, the preimage of which is composed of two intersecting lines, precisely $S(1)$ and a line of the ruling. To show this we prove the following stronger claim.

CLAIM. Let $\alpha = \langle S(1), P\rangle = \langle \ell, P\rangle$ be the plane spanned by $S(1)$, or by ℓ, and the point P. Then the intersection cycle $v(\alpha, S(1,4))$ is composed by the union of $S(1)$ and a line, say r, of the ruling of $S(1,4)$ and three movable points on $S(1)$. In particular, $\pi_P^{-1}(\ell) = \alpha \cap S(1,4) = S(1) \cup r$.

Proof. To prove the claim, let $H \in \mathbb{P}^5$ be a generic hyperplane containing α. The hyperplane H intersects $S(1,4)$ in a curve which is a union of lines and having $S(1)$ as a component. In fact, H intersects $S(4)$ in four distinct points $Q_1,\dots,Q_4$ and through each of them there is a line of the ruling lying on H and intersecting $S(1)$ in a point. Denote these distinct lines by $r_1,\dots,r_4$ and let $R_i = r_i \cap S(1)$.

We observe that exactly one of the lines $r_1,\dots,r_4$ is contained in α. In fact, being $P \in X$, there exists a line $l \subset X$ passing through P. Such a line is contained in α, but it cannot be a line on $S(1,4)$ since $P \notin S(1,4)$. Let $Q \in (S(1,4) \setminus S(1)) \cap l$. Such a point Q is the only point in which the plane α intersects $S(4)$, since a plane can contain only one line of the ruling. Let $r = r_1$ be the unique line of $S(1,4)$ passing through $Q = Q_1$. This line r is contained in α since the point Q and the point $r \cap S(1)$ are in α, hence $S(1,4) \cap \alpha = S(1) \cup r$.

The intersection cycle $v(\alpha, S(1,4))$ is composed by $r \cup S(1)$ and the three embedded points R_2, R_3, R_4, which are movable on $S(1)$ when $H(\supset \alpha)$ varies. $\square$

Let $P = [0:1:1:0:0:0:0] \in X \setminus S(1,4)$ and let π_P be the linear projection from P into $\mathbb{P}^5 = \mathrm{Proj}(\mathbb{C}[x_0, x_1 - x_2, x_3, x_4, x_5, x_6])$. The surface $Z := Z(1,4) := \pi_P(S(1,4))$ is defined by the ideal

$$\begin{aligned}&(-x_3x_5 + x_4{}^2, -x_3x_6 + x_4x_5, -x_4x_6 + x_5{}^2,\\ &-x_0x_5 + (x_2 - x_1)x_4 + x_3{}^2, -x_0x_6 + (x_2 - x_1)x_5 + x_3x_4).\end{aligned}$$

It has degree 5 and it is singular along the line of equations $x_3 = x_4 = x_5 = x_6 = 0$, which is a line of double points, whose preimage is composed by the two intersecting lines of equations $x_2 = \cdots = x_6 = 0$ and $x_1 = x_3 = \cdots = x_6 = 0$ respectively.

The surface Z is a Verra variety since if we project the surface from ℓ, its image is a rational normal cubic curve, hence Z is given by Verra's construction.

One can observe that the surface Z has the same intersection numbers of the surface $T(2,3)$:

$$c(Z,Z) = (2,8,10,5) = 2(1,0,0,0) + 2(0,2,1,0) + (0,4,8,5).$$

In Examples 1, 4 and 5 we considered surfaces with one apparent double point. Recently Ciliberto and Russo [11] gave a complete classification of (possibly singular) surfaces $S \subset \mathbb{P}^5$ with one apparent double point, proving that S is either a smooth rational normal scroll or a (weak) del Pezzo surface of degree 5 or a Verra surface constructed from a rational normal cubic.

EXAMPLE 6. *Surface in $\mathbb{P}^5$ with a line of double points, an isolated singular point which contributes to the self intersection cycle and two apparent double points*
Let us consider the rational normal scroll

$$S(4,2) \subset \mathbb{P}^7 = \mathrm{Proj}(\mathbb{C}[x_0,\dots,x_4,y_0,y_1,y_2]).$$

The ideal of its tangent variety is

$$
\begin{gathered}
(x_0x_4-4x_1x_3+3x_2^2,\quad 4x_0x_2x_4-3x_0x_3^2-3x_1^2x_4+2x_1x_2x_3,\\
-4x_0^2x_4^2+14x_0x_1x_3x_4-9x_0x_2x_3^2-9x_1^2x_2x_4+8x_1^2x_3^2,\\
-3x_0x_2y_2+3x_0x_3y_1-x_0x_4y_0+3x_1^2y_2-3x_1x_2y_1+x_1x_3y_0,\\
-3x_0x_3y_2+4x_0x_4y_1+3x_1x_2y_2-4x_1x_3y_1-3x_1x_4y_0+3x_2x_3y_0,\\
-x_0x_4y_2+x_1x_3y_2+3x_1x_4y_1-3x_2x_3y_1-3x_2x_4y_0+3x_3^2y_0,\\
x_0y_2^2-4x_1y_1y_2+2x_2y_0y_2+4x_2y_1^2-4x_3y_0y_1+x_4y_0^2,\\
x_0x_1x_4y_2-9x_0x_2x_3y_2+12x_0x_2x_4y_1-3x_0x_3x_4y_0+8x_1^2x_3y_2-12x_1^2x_4y_1+3x_1x_2x_4y_0,\\
3x_0^2x_4y_2-9x_0x_1x_3y_2-x_0x_1x_4y_1+9x_0x_2x_3y_1-3x_0x_2x_4y_0+6x_1^2x_2y_2-8x_1^2x_3y_1+3x_1^2x_4y_0).
\end{gathered}
$$

We observe that the point $Q=[0:\ldots:0:1:0]\in \operatorname{Tan}S(4,2)$ has multiplicity two for $\operatorname{Tan}S(4,2)$.

Let us project $S(4,2)$ from the line passing through the points

$$P=[1:0:0:0:1:0:0:0]\in \operatorname{Sec}S(4,2)\setminus \operatorname{Tan}S(4,2)$$

and

$$Q=[0:0:0:0:0:0:1:0]\in \operatorname{Tan}S(4,2),$$

that is, from the line

$$L=\{[x_0:\ldots:x_4:y_0:y_1:y_2]\in\mathbb{P}^7\mid x_0-x_4=x_1=x_2=x_3=y_0=y_2=0\},$$

not contained in $\operatorname{Sec}S(4,2)$ and intersecting $\operatorname{Tan}S(4,2)$ only in the point Q. The projection $S=\pi_L(S(4,2))\subset\mathbb{P}^5=\{[x_0:\ldots:x_4:y_0:y_1:y_2]\in\mathbb{P}^7\mid x_0-x_4=y_1=0\}$ is a singular surface with a line of double points and an isolated double point. After a change of coordinates (in which we eliminate x_0-x_4 and y_1), in the new coordinates we have

$$\operatorname{Sing}S=\{[x_1:\ldots:x_4:y_0:y_2]\in\mathbb{P}^5\mid x_1=x_2=x_3=x_4=0\}\cup\{[1:0:0:0:0:0]\}=\ell\cup R$$

and

$$
\begin{aligned}
\operatorname{Tan}S= \quad & \{[x_1:\ldots:x_4:y_0:y_2]\in\mathbb{P}^5\mid\\
- \quad & x_1^2x_2^2x_4^2+6x_1x_2^3x_3x_4-4x_1x_2^2x_3^3-6x_1x_2x_3x_4^3+4x_1x_3^3x_4^2-4x_2^5x_4\\
+ \quad & 3x_2^4x_3^2-8x_2^3x_4^3+42x_2^2x_3^2x_4^2-48x_2x_3^4x_4-4x_2x_4^5+16x_3^6+3x_3^2x_4^4=0\}.
\end{aligned}
$$

The surface S is a Verra variety since if we project the surface from ℓ, its image is a quartic curve C with self-intersection vector

$$c(C,C)=(4,8,4)=4(1,0,0)+2(0,1,0)+(0,6,4),$$

hence $\rho(S)=\rho(C)=2$. This can also be confirmed by the computation of $c(S,S)=(4,14,12,6)$.

The singular point $R\in S$ contributes to the cycle with multiplicity $j=2$, since in the affine chart $x_1=1$ the self-intersection vector of S is $(2,0,2)$.

EXAMPLE 7. *Surface in* $\mathbb{P}^5$ *with two lines of double points and two apparent double points*
Let us consider the rational normal scroll $S(4,2) \subset \mathbb{P}^7$ and project it from the line

$$s = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_2 = x_3 = x_4 = x_5 = x_7 = 0\}$$

on the linear space $\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_6 = 0\} \cong \mathbb{P}^5$. The line s is contained in $\operatorname{Tan} S(4,2)$ and intersects $\operatorname{Sing}(\operatorname{Tan} S(4,2))$ in a point.

We obtain a surface $T \subset \mathbb{P}^5$ with two lines of double points

$$r_1 : \{[x_0 : x_2 : x_3 : x_4 : x_5 : x_7] \in \mathbb{P}^5 \mid x_2 = x_3 = x_4 = x_7 = 0\},$$

$$r_2 : \{[x_0 : x_2 : x_3 : x_4 : x_5 : x_7] \in \mathbb{P}^5 \mid x_0 = x_2 = x_3 = x_4 = 0\},$$

intersecting in the point $P = r_1 \cap r_2 = [0 : \ldots : 0 : 1 : 0 : 0]$ and such that the preimage of r_1 in the projection is the line

$$\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_1 = x_2 = x_3 = x_4 = x_6 = x_7 = 0\}$$

on the scroll and the preimage of r_2 in the projection is the conic

$$\{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 = x_2 = x_3 = x_4 = x_6^2 - x_5x_7 = 0\}.$$

The intersection vector of T is

$$c(T,T) = (4,12,14,6) = 4(1,0,0,0) + \{3(0,1,1,0) + 2(0,2,1,0)\} + (0,5,9,6),$$

where the contribution inside the curly braces refers to the one-dimensional part of the singular locus. Precisely $j_{r_1} = 3$ and there is a movable point of multiplicity 3 on r_1, $j_{r_2} = 2$ and there are two movable points of multiplicity 2 on r_2 .

The surface T is a Verra variety since if we project it from r_2, its image is a quartic curve C with self-intersection vector $c(C,C) = (4,8,4)$, hence $\rho(T) = \rho(C) = 2$.

We observe that $c(S(4,2),S(4,2)) = (12,8,10,6)$, and this shows that

$$\deg P_1(S(4,2)) > \deg P_1(T).$$

4. Computing ρ of a projection

In this section we propose a second method for the computation of $\rho(T)$ when the singular surface $T \subset \mathbb{P}^5$ arises as the projection of a surface $S \subset \mathbb{P}^N_K$ $(N > 5)$ along a linear subspace L. This method reduces considerably the computational cost when calculating concrete examples (see Section 5).

THEOREM 3. *Let* $X \subset \mathbb{P}^N$ *be a non-degenerate reduced and irreducible variety such that* $2\dim X + 1 < N$ *and the generic tangent hyperplane to* $\operatorname{Sec} X$ *is tangent to* X *at only finitely many points (that is,* X *is not* 1*-weakly defective in the sense of [9],* *2.1). Let $L \subset \mathbb{P}^N$ be a linear subspace such that* $\operatorname{codim} L > \dim X + 1$ *and such that the linear projection $\pi_L : X \to \pi_L(X) =: Y$ is generically one to one.*

Then

$$\rho(Y) \cdot \deg \operatorname{Sec} Y = c_0(L, \operatorname{Sec} X),$$

and, if L is a point and $e(\operatorname{Sec} X, L)$ denotes the multiplicity of $\operatorname{Sec} X$ *at L (which is defined to be zero if $L \notin \operatorname{Sec} X$), it holds*

$$\rho(Y) \cdot \deg \operatorname{Sec} Y = \deg \operatorname{Sec} X - e(\operatorname{Sec} X, L).$$

Proof. It is known that

$$\rho(Y) = \frac{1}{2} \deg(J(Y,Y)/\operatorname{Sec} Y)$$

and $\deg(J(X,X)/\operatorname{Sec} X) = 2$ (see Proposition 2 and Remark 1). Let us consider the following diagram of rational maps

$$\begin{array}{ccc} J(X,X) & \xrightarrow{\pi_{\Delta^N}} & \operatorname{Sec} X \\ \Big\downarrow {\scriptstyle J(\pi_L,\pi_L)} & & \Big\downarrow {\scriptstyle \pi_L} \\ J(Y,Y) & \xrightarrow{\pi_{\Delta^{N-k-1}}} & \operatorname{Sec} Y \end{array}$$

where $J(\pi_L, \pi_L)$ is the map induced by π_L, π_{Δ^N} and $\pi_{\Delta^{N-k-1}}$ are the projections along the diagonal spaces of $J(\mathbb{P}^N, \mathbb{P}^N)$ and $J(\mathbb{P}^{N-k-1}, \mathbb{P}^{N-k-1})$, respectively. By assumption $\deg J(\pi_L, \pi_L) = 1$, hence by the commutativity of the diagram it turns out that

$$\deg(J(Y,Y)/\operatorname{Sec} Y) = 2 \deg(\operatorname{Sec} X/\operatorname{Sec} Y),$$

hence

$$\rho(Y) = \deg(\operatorname{Sec} X/\operatorname{Sec} Y). \tag{5}$$

On the other hand, by van Gastel [19]

$$c_0(L, \operatorname{Sec} X) = \deg(J(L, \operatorname{Sec} X)/\mathrm{embJ}(L, \operatorname{Sec} X)) \cdot \deg(\mathrm{embJ}(L, \operatorname{Sec} X)).$$

Since one of the two intersecting varieties is a linear space, the cycle $v(L, \operatorname{Sec} X)$ can be computed without passing to the ruled join (see [16], Proposition 2.2.11), therefore

$$\begin{aligned} c_0(L, \operatorname{Sec} X) &= \deg(\operatorname{Sec} X/\pi_L(\operatorname{Sec} X)) \cdot \deg \pi_L(\operatorname{Sec} X) \\ &= \deg(\operatorname{Sec} X/\operatorname{Sec} Y) \cdot \deg \operatorname{Sec} Y, \end{aligned}$$

which, together with (5), finishes the proof of the first formula.

If L is a point, by the refined Theorem of Bezout and taking into account that $j(\operatorname{Sec} X, L; L) = e(\operatorname{Sec} X, L)$ (see e.g. [16], Lemma 5.4.7) one has

$$c_0(L, \operatorname{Sec} X) = \deg \operatorname{Sec} X - e(\operatorname{Sec} X, L),$$

which finishes the proof in this case. □

COROLLARY 1. *Let $S \subset \mathbb{P}^N, N > 5$ be a non-degenerate reduced and irreducible surface such that the generic tangent hyperplane to* $\operatorname{Sec} S$ *is tangent to S at only finitely many points (that is, S is not* 1*-weakly defective in the sense of [9], 2.1). Let $L \subset \mathbb{P}^N$ be a linear subspace of codimension* 6 *such that the linear projection $\pi_L \colon S \to \pi_L(S) =: T \subset \mathbb{P}^5$ is generically one to one.*

Then $\rho(T) = c_0(L, \operatorname{Sec} S)$.

Using this result we can compute the number ρ for some surfaces (of low degree) in a class of Verra surfaces in $\mathbb{P}^5$ with a multiple line and one apparent double point, which contains the surface of Example 5.

EXAMPLE 8. Let us consider the rational normal scroll $S(1,d-1)$ of degree d in $\mathbb{P}^{d+1}$, with $d \geq 5$. Let $\Lambda \subset \mathbb{P}^{d+1}$ be a linear subspace of dimension $d-5$ such that

$$\Lambda \cap S(1,d-1) = \emptyset, \quad \Lambda \cap \operatorname{embJ}(S(1), S(d-1)) = \{P_1, \ldots, P_k\}, \, 1 \leq k \leq d-4.$$

We remark that $\operatorname{Sec} S(1,d-1)$ (resp. $\operatorname{Tan} S(1,d-1)$) is a cone of vertex $S(1)$ over $\operatorname{Sec} S(d-1)$ (resp. $\operatorname{Tan} S(d-1)$) and that $\operatorname{Sing}\operatorname{Sec} S(1,d-1)$ (resp. $\operatorname{Sing}\operatorname{Tan} S(1,d-1)$) is a cone of vertex $S(1)$ over $\operatorname{Sing}\operatorname{Sec} S(d-1) = S(d-1)$ (resp. $\operatorname{Sing}\operatorname{Tan} S(d-1) = S(d-1)$), hence

$$\begin{aligned} \operatorname{embJ}(S(1), S(1,d-1)) &= \operatorname{embJ}(S(1), S(d-1)) = \\ &= \operatorname{Sing}\operatorname{Sec} S(1,d-1) = \operatorname{Sing}\operatorname{Tan}(S(1,d-1)). \end{aligned}$$

Moreover, $d-4$ is the maximum number of points of $\Lambda \cap \operatorname{embJ}(S(1), S(d-1))$. In fact if Λ would intersect $\operatorname{embJ}(S(1), S(d-1))$ in $m > d-4$ points, through each of them there would be a line l_i connecting $S(1)$ with $S(d-1)$. Let $Q_i = l_i \cap S(d-1)$ and let r_i be the line of the ruling through Q_i. We observe that the point

$$Q_i \in \alpha = \langle \Lambda, S(1) \rangle \cong \mathbb{P}^{d-3},$$

hence the line r_i is contained in α. Since $S(d-1)$ is not contained in α, starting from a point $Q_{m+1} \in S(d-1)$ we can find a line r_{m+1} of the ruling which is not contained in α. The linear space

$$\langle \alpha, r_{m+1} \rangle \cong \mathbb{P}^{d-2}$$

contains $m+1 > d-3$ lines of the ruling and repeating this reasoning we could find a hyperplane $H \cong \mathbb{P}^d$ containing d lines of the ruling and the line $S(1)$ and this contradicts the theorem of Bezout, since the scroll $S(1,d-1)$ is a non degenerate surface.

The intersection cycle $v(S(1,d-1), \alpha)$ is composed by the lines $S(1), r_1, \ldots, r_k$ and $d-1-k$ movable points on $S(1)$.

Now let $Z(1,d-1) = \pi_\Lambda(S(1,d-1)) \subset \mathbb{P}^5$. Such surface has a singular line, say ℓ, of multiplicity $k+1$ whose preimage are the lines $S(1), r_1, \ldots, r_k$. We can project the surface Z from ℓ into $\mathbb{P}^3$ and we obtain an irreducible curve C of degree $d-k-1$ and Z turns out to be a Verra surface constructed from ℓ and C, hence $\rho(Z) = \rho(C)$.

If $k = d-4$ then $\rho(Z) = \rho(C) = 1$, if $k < d-4$ we can compute $\rho(C)$ by computing the self-intersection cycle of C (see [4]) or using Corollary 1.

Now we show the application of Corollary 1 to projections of $S(1,5)$ and $S(1,6)$ into $\mathbb{P}^5$.

EXAMPLE 9. Let us project the rational normal scroll $S(1,5) \subset \mathbb{P}^7$ into $\mathbb{P}^5$ from the line

$$s_1 = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 = 0\},$$

which intersects $\mathrm{Sec}\, S(1,5)$ in the point

$$P = [0 : 1 : 1 : 0 : \ldots : 0] \in \mathrm{Sing}(\mathrm{Sec}\, S(1,5))$$

of multiplicity 3 for $\mathrm{Sec}\, S(1,5)$, that is $k = 1$. The image is the surface $Z_1 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_1 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_7]/I(S(1,5)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7.$$

The singular locus of Z_1 is the double line

$$\ell_1 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}.$$

We have

$$c(\mathrm{Sec}\, S(1,5), s_1) = (2,4) = (2,0) + 2(0,1) + (0,2),$$

in particular $\rho(Z_1) = 2$. Here $2(0,1) + (0,2)$ is the contribution of P, which comes from two components of the normal cone to $s_1 \cap \mathrm{Sec}\, S(1,5)$ in $\mathrm{Sec}\, S(1,5)$, therefore $j(\mathrm{Sec}\, S(1,5), s_1; P) = 4$.

If we project now $S(1,5)$ into $\mathbb{P}^5$ from the line

$$s_2 = \{[x_0 : \ldots : x_7] \in \mathbb{P}^7 \mid x_0 = x_1 - x_2 = x_3 = x_4 = x_5 = x_6 - x_7 = 0\},$$

which intersects $\mathrm{Sec}\, S(1,5)$ in the same point

$$P = [0 : 1 : 1 : 0 : \ldots : 0] \in \mathrm{Sing}(\mathrm{Sec}\, S(1,5))$$

as before and in the smooth point $Q = [0 : \ldots : 0 : 1 : 1]$ (that is $k = 1$), we obtain a surface $Z_2 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_2 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_7]/I(S(1,5)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3,\ z_3 \mapsto x_4,\ z_4 \mapsto x_5,\ z_5 \mapsto x_6 - x_7.$$

The singular locus of Z_2 is composed of the double line

$$\ell_2 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}$$

and the isolated point $[0 : \ldots : 0 : 1]$. We have

$$c(\mathrm{Sec}\, S(1,5), s_2) = (2,4) = 2(1,0) + (0,3) + (0,1),$$

in particular $\rho(Z_2) = 2$, $j(\operatorname{Sec} S(1,5), s_2; P) = 3$ and $j(\operatorname{Sec} S(1,5), s_2; Q) = 1$.

Both Z_1 and Z_2 are Verra surfaces constructed from ℓ_i ($i = 1,2$) and the irreducible quartic curve $C_i \subset \mathbb{P}^3$ which is the projection of Z_i from ℓ_i. The curve C_i has a double point and the self-intersection vector of C_i is $c(C_i, C_i) = (4,8,4)$, in particular $\rho(C_i) = 2$.

EXAMPLE 10. Let us project the rational normal scroll $S(1,6) \subset \mathbb{P}^8$ into $\mathbb{P}^5$ from the plane

$$\pi_1 = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 - x_8 = 0\},$$

which intersects $\operatorname{Sec} S(1,6)$ in the point

$$P = [0:1:1:0:\ldots:0] \in \operatorname{Sing}(\operatorname{Sec} S(1,6))$$

of multiplicity 4 on $\operatorname{Sec} S(1,6)$ and in the smooth point $Q = [0:\ldots:0:1:1]$, hence $k = 1$. The image is the surface $Z_3 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_3 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_8]/I(S(1,6)),$$

$$z_0 \mapsto x_0,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7 - x_8.$$

The singular locus of Z_3 is the double line

$$\ell_3 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}$$

and the point $[0:\ldots:0:1]$. We have

$$c(\operatorname{Sec} S(1,6), \pi_1) = (4,6) = 4(1,0) + 2(0,1) + (0,3) + (0,1),$$

in particular $\rho(Z_3) = 4$. Here $2(0,1) + (0,3)$ is the contribution of P and $(0,1)$ is the contribution of Q, that is, $j(\operatorname{Sec} S(1,6), \pi_1; P) = 5$ and $j(\operatorname{Sec} S(1,6), \pi_1; Q) = 1$.

Now let us project $S(1,6)$ into $\mathbb{P}^5$ from the plane

$$\pi_2 = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 - x_8 = x_1 - x_2 = x_3 - x_4 = x_5 = x_6 = x_7 = 0\},$$

which intersects $\operatorname{Sec} S(1,6)$ in the line

$$\ell = \{[x_0 : \ldots : x_8] \in \mathbb{P}^8 \mid x_0 - x_8 = x_1 - x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = 0\}$$

and $\operatorname{Sing}(\operatorname{Sec} S(1,6))$ in the two points $P_1 = [0:1:1:0:\ldots:0]$ and $P_2 = [1:0:\ldots:0:1]$ of multiplicity 4 on $\operatorname{Sec} S(1,6)$, hence $k = 2$. The image is the surface $Z_4 \subset \mathbb{P}^5$ defined by the kernel of the map

$$\phi_4 \colon K[z_0, \ldots, z_5] \to K[x_0, \ldots, x_8]/I(S(1,6)),$$

$$z_0 \mapsto x_0 - x_8,\ z_1 \mapsto x_1 - x_2,\ z_2 \mapsto x_3 - x_4,\ z_3 \mapsto x_5,\ z_4 \mapsto x_6,\ z_5 \mapsto x_7.$$

The singular locus of Z_4 is the triple line

$$\ell_4 = \{[z_0 : \ldots : z_5] \in \mathbb{P}^5 \mid z_2 = z_3 = z_4 = z_5 = 0\}.$$

We have

$$c(\operatorname{Sec} S(1,6), \pi_2) = (2,7,1) = 2(1,0,0) + 2(0,1,0) + (0,2,0) + (0,3,0) + (0,0,1),$$

in particular $\rho(Z_4) = 2$. Here $2(0,1,0) + (0,2,0)$ is the contribution of P_1, $(0,3,0)$ is that of P_2, and $(0,0,1)$ is the contribution of ℓ, that is, $j(\operatorname{Sec} S(1,6), \pi_2; P_1) = 4$, $j(\operatorname{Sec} S(1,6), \pi_2; P_2) = 3$ and $j(\operatorname{Sec} S(1,6), \pi_1; \ell) = 1$.

Both Z_3 and Z_4 are Verra surfaces constructed from ℓ_i ($i = 3,4$) and the irreducible curve $C_i \subset \mathbb{P}^3$ which is the projection of Z_i from ℓ_i. The curve C_3 has degree 5 and two double points with $j = 3$, moreover its self-intersection vector is $c(C_3, C_3) = (8,12,5)$, in particular $\rho(C_3) = 4$. The curve C_4 has degree 4 and a double point, say R, such that $j(C_4, C_4; R) = 3$. The self-intersection vector of C_4 is $c(C_4, C_4) = (4,8,4)$, in particular $\rho(C_4) = 2$.

5. Code of procedures for computing the examples

Code for computing the bidegrees $c_k(\mathcal{R})$ of a bigraded ring $\mathcal{R}$

While REDUCE (using the package SEGRE [1]) and Macaulay2 provide the functions `degs` or `multideg` and `multidegree` respectively, in CoCoA and Singular the bidegrees $c_k(\mathcal{R})$ must be computed from the numerator of the non-simplified Hilbert series of $\mathcal{R}$ according to [25], p. 167. Furthermore, in CoCoA 4.7.4 it is not possible to assign bidegrees beginning with zeros as $\deg(t_i) = (0,1)$. The trick is to pass to a $\mathbb{Z}^3$-graded ring with $\deg(x_i) = \deg(y_i) = (1,1,0)$ and $\deg(t_i) = (1,0,1)$. For the convenience of the reader we provide the details:

CODE 1. *CoCoA, version 4.7.4*:

```
Define BiDegree(I,A,B)

  F := Flatten([[1 | X In 1..A],[0 | X In 1..B]]);

  S := Flatten([[0 | X In 1..A],[1 | X In 1..B]]);

  G:= Mat([[1 | X In 1..(A+B)], F, S]);
```

This creates the matrix

$$G = \begin{pmatrix} 1\ldots\ldots1 & 1\ldots\ldots1 \\ 1\ldots\ldots1 & 0\ldots\ldots0 \\ \underbrace{0\ldots\ldots0}_{A\text{ columns}} & \underbrace{1\ldots\ldots1}_{B\text{ columns}} \end{pmatrix}$$

the columns of which are the degrees of the $A+B$ ring variables for the subsequent Hilbert series computation. Note that since CoCoA-4 does not allow zero-entries in the first row of the matrix which defines the degrees, we have added a first row of ones.

```
    H := HilbertSeriesMultiDeg(CurrentRing()/I, G);
```

Now we extract the numerator of the non-simplified Hilbert series:

```
    Num := Sum([X[1]*LogToTerm(X[2]) | X In @H[1]]);
```

According to [25], p. 167, the normalized leading coefficients of the Hilbert polynomial are obtained from the numerator `Num` by substituting each variable t by $1-t$ and then collecting the coefficients of the terms having total degree codim(CurrentRing$/I$), i.e., the coefficients of the lowest degree terms. To get rid of the artificially introduced first variable which is due to the first row of G, this variable must be substituted by one. Doing this, we obtain from the numerator the polynomial N, which we write as a polynomial in the first two variables of the current ring.

```
    N := Eval(Num, [1, 1-Indet(1), 1-Indet(2)]);

    M := Min([Deg(X) | X In Monomials(N)]);

    P := Sum([X In Monomials(N) | Deg(X) = M]);
```

The polynomial P, written in the first two variables of the current ring, is the bidegree in the sense of [25], p. 167. For better readability, the coefficients of P are printed, but P is returned:

```
    PrintLn [CoeffOfTerm(X,P) | X In Support((Indet(1)+Indet(2))^M)];

  Return P;

  EndDefine;
```

Singular, version 3.1.6:

We give the code of a procedure which computes the bidegrees of an ideal I.

```
LIB "multigrading.lib";

proc bidegree(ideal I, int a, int b)

{

  ideal SI = std(I);

  def currentring = basering;

  int n = nvars(basering);
```

```
intmat m[2][a+b] = 1:a,0:b,0:a,1:b;
```

This creates the matrix

$$m = \begin{pmatrix} 1\ldots\ldots 1 & 0\ldots\ldots 0 \\ \underbrace{0\ldots\ldots 0}_{a \text{ columns}} & \underbrace{1\ldots\ldots 1}_{b \text{ columns}} \end{pmatrix}$$

the columns of which are the degrees of the $a+b$ ring variables.

```
setBaseMultigrading(m);

def h = hilbertSeries(SI);

setring h;

poly f = substitute(numerator1,t_(1),1-t_(1),t_(2),1-t_(2));
```

Here `numerator1` is the numerator of the non-simplified Hilbert series, which is called the *first Hilbert series* in the Singular Manual.

```
poly g = jet(f,mindeg(f));
```

The polynomial g, that is, the homogeneous part of lowest degree of f, is by [25] the bidegree of I. It will be returned as a polynomial of the base ring written in the first two variables of the base ring.

```
  setring currentring;

  return(fetch(h,g));

};
```

Code for computing the defining ideals of rational normal scrolls $S(a,b)$

The defining ideals of $S(a,b)$ can be conveniently computed by various computer algebra systems, e.g. using the following functions:

CODE 2. *Macaulay2, version 1.4:*

```
scroll = (a,b,K) ->

(

R := K[x_0 .. x_a, y_0 .. y_b];
```

```
M := map(R^2, a, (i,j)->x_(i+j));

N := map(R^2, b, (i,j)->y_(i+j));

I := minors(2, M|N)

)
```

CoCoA, version 4.7.4:

```
Define Scroll(A,B)

  ScrollRing ::=  Q[x[0..A],y[0..B]];

  Using ScrollRing Do M := Mat([Concat(x[0]..x[A-1],y[0]..y[B-1]),

  Concat(x[1]..x[A],y[1]..y[B])]);

  Return Ideal(Minors(2,M));

  EndUsing;

EndDefine;
```

Singular, version 3.1.6:

```
proc scroll(int a, int b, int ch)

  {

  ring scrollring = ch,(x(0..a),y(0..b)),dp;

  matrix M[2][a+b] = x(0..a-1),y(0..b-1),x(1..a),y(1..b);

  ideal scrollideal = minor(M,2);

  export(scrollring,scrollideal);

  }
```

Code for computing the intersection vector

We refer to Example 3 in order to explain the code we used for the computer aided calculations in our examples.

CODE 3. REDUCE:

```
Reduce (Free PSL version), 30-Nov-11 ...

1: load_package segre;

SEGRE 1999/2012-07-11 with package CALI, for help type: help(help);

2: setideal(s33, scroll{3,3})$

3: t := eliminate(s33, {x1,x6})$

4: setring({x0,x2,x3,x4,x5,x7},{},lex)$

5: setideal(nc, int_ncone{t,t})$

6: degs(nc, {6,6});

{8,12,10,6,0,0,0}

7: on time;

Time: 17284 ms  plus GC time: 579 ms
```

Macaulay2, version 1.4:

```
i1 : load "scroll.m2"

i2 : t1 = cpuTime();

i3 : S33 = scroll(3,3,QQ);

o3 : Ideal of QQ[x , x , x , x , y , y , y , y ]
                  0   1   2   3   0   1   2   3

i4 : ringP7 = ring(S33);

i5 : ringP5 = QQ[z_0 .. z_5];

i6 : center = {x_0, x_2, x_3, y_0, y_1, y_3}

o6 = {x , x , x , y , y , y }
       0   2   3   0   1   3

o6 : List

i7 : T = trim kernel map(ringP7/S33, ringP5, center);

o7 : Ideal of ringP5
```

```
i8 : idealNormalCone = intNcone(T,T);

o8 : Ideal of QQ[z , z , z , z , z , z , w , w , w , w , w , w ]
                  0   1   2   3   4   5   0   1   2   3   4   5

i9 : multidegree idealNormalCone

       6      5        4 2     3 3
o9 = 8T  + 12T T  + 10T T  + 6T T
       0      0 1      0 1     0 1

o9 : ZZ[T , T ]
         0   1

i10 : cpuTime() - t1  --time in ms, CPU Intel(R) Core(TM) i5-2410M

o10 = 395.618

o10 : RR (of precision 53)
```

CoCoA, version 4.7.4:

```
Source "scroll.coc";
Set Timer;
Null
-------------------------------
S33:=Scroll(3,3);

Cpu time = 0.31, User time = 0
-------------------------------
Use ScrollRing;

Cpu time = 0.00, User time = 0
-------------------------------
T:=Elim([x[1],y[2]],S33);

Cpu time = 0.62, User time = 0
-------------------------------
Use RingP5::=Q[z[0..5]];

Cpu time = 0.00, User time = 0
-------------------------------
F:=RMap(z[0],0,z[1],z[2],z[3],z[4],0,z[5]);

Cpu time = 0.00, User time = 0
-------------------------------
T:=Ideal(Image(Gens(T),F));
```

```
Cpu time = 0.47, User time = 0
-------------------------------
J:=RuledJoin(T,T);

Cpu time = 0.31, User time = 0
-------------------------------
Use JoinRing;

Cpu time = 0.00, User time = 0
-------------------------------
B:=BlowUp(J[1],J[2]);

Cpu time = 31.52, User time = 3
-------------------------------
N:=NumIndets(BlowUpRing)/3;

Cpu time = 0.00, User time = 0
-------------------------------
Use CoeffRing[x[1..N],t[1..N]];

Cpu time = 0.00, User time = 0
-------------------------------
G:=RMap(Concat(x[1]..x[N],x[1]..x[N],t[1]..t[N]));

Cpu time = 0.00, User time = 0
-------------------------------
NormalCone:=Image(B[2],G);

Cpu time = 1.09, User time = 0
-------------------------------
BiDegree(NormalCone,6,6);
[8, 12, 10, 6, 0, 0, 0]
8x[1]^6 + 12x[1]^5x[2] + 10x[1]^4x[2]^2 + 6x[1]^3x[2]^3
-------------------------------

Cpu time = 1.25, User time = 0
-------------------------------
```

Singular, version 3.1.6, input file:

```
< "scroll.s";

timer = 0;

system("--ticks-per-sec",1000);

int t1 = timer;
```

```
scroll(3,3,0);

ideal s33 = scrollideal;

ideal t = eliminate(s33,x(1)*y(2));

ring ringP5 = 0, (x(0),x(2),x(3),y(0),y(1),y(3)), dp;

ideal t = imap(scrollring, t);

rjoin(t,t);

setring joinring;

formring(joinideal, diagonalideal);

int n = nvars(form_r)/3;

ring R = char(form_r),(x(1..n),t(1..n)),dp;

setring R;

map f = form_r, x(1..n),x(1..n),t(1..n);

bidegree(f(form_i),6,6);

"time in ms = ", timer-t1;

quit;
```

Output file (running Singular in quite mode):

```
Singular -q < inputfile > outputfile
```

```
 ------------
This proc returns a ring with polynomials called 'numerator1/2'
and 'denominator1/2'!
They represent the first and the second Hilbert Series.
The s_(i)-variables are defined to be the inverse of the
t_(i)-variables.
 ------------
8*x(1)^6+12*x(1)^5*x(2)+10*x(1)^4*x(2)^2+6*x(1)^3*x(2)^3
time in ms =  10300
```

Code for computing the secant variety

If $T = \pi_L(S)$ is a linear projection of S from L, then Theorem 3 and Corollary 1 can be applied to compute $\rho(T)$ by computing $c_0(\operatorname{Sec} S, L)$. Hence the defining ideal of the secant variety $\operatorname{Sec} S$ has to be computed. Nevertheless this method reduces the computation time of $T = \pi_L(S)$ considerably.

For example, the times required for the computation of Example 3 (see the previous subsection) reduces from ca. 18 seconds to less than 1 second (REDUCE), from 4 seconds to less than 1 second (CoCoA), from more than 6 minutes to 2 seconds (Macaulay2), and from ca. 11 seconds to less than 1 second (Singular). The computations have been performed using a Cygwin installation under Microsoft Windows 7 with CPU Intel(R) Core(TM) i5-2410M.

REDUCE (with the package SEGRE) has the built-in facilities `ej(I,J)` and `ejoin({I,J})` which permit the calculation of the ideal of the embedded join the projective varieties defined by the homogeneous ideals I and J. If $I = J$, then this is the ideal of the secant variety of the projective variety defined by I.

Here we propose analogue procedures for CoCoA, Macaulay2, and Singular.

CODE 4. *Macaulay2, version 1.4:*

```
embJoin = (I,J) ->

(

R := ring(I);

K := coefficientRing(R);

n := numgens(R);

T := tensor(R/I,R/J);

G := gens(T);

x := take(G,{0,n-1});

y := take(G,{n,2*n-1});

F := map(T,R,x-y);

ker F

)
```

Singular, version 3.1.6:

```
// Author: Peter Schenzel, schenzel@informatik.uni-halle.de
```

```
proc join(ideal I, ideal J)
{
  def rj = basering;
  int n = nvars(rj);
  def sj = extendring(n,"v(","c,dp",1,rj);
  setring sj;
  ideal I1 = imap(rj,I);
  ideal J1 = imap(rj,J);
  int j;
  for(j = 1; j <= n; j++)
    {
      I1 = subst(I1,var(j),v(j));
      J1 = subst(J1,var(j),var(j)-v(j));
    }
  ideal K = I1+J1;
  ideal join = elim(K,n+1..2*n);
  setring rj;
  ideal join = imap(sj,join);
  return(join);
};
```

Further procedures needed include those for associated graded rings of quotient rings, which can be obtained by standard elimination theory. They are built-in functions in SEGRE and Macaulay2 but not in CoCoA and Singular. We shall not reproduce here our code (which is certainly not optimal), but make it available at http://www.dm.unibo.it/∼achilles/code.

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