C.d.L. in Scienze naturali – Matematica 17. 12. 2015

1. Dati la matrice $\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}$ e il vettore $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, calcolare $\mathbf{v}^T\mathbf{A}\mathbf{v}$, dove $\mathbf{v}^T = [x, y]$ è il cosiddetto trasposto di $\mathbf{v}$.

   $$\mathbf{v}^T\mathbf{A}\mathbf{v} = [x, y]\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x, y]\begin{bmatrix} 5x - y \\ -x + 3y \end{bmatrix} = 5x^2 - 2xy + 3y^2$$

2. Risolvere il seguente sistema lineare con l'algoritmo di Gauss-Jordan:

$$\begin{cases} x_1 - 2x_2 + 2x_3 = 8 \\ -2x_1 + 3x_2 - 5x_3 = 0 \\ x_1 + 2x_3 = -3 \end{cases}.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ -2 & 3 & -5 & 0 \\ 1 & 0 & 2 & -3 \end{array}\right] \begin{array}{l} R2 + 2R1 \rightarrow \\ R3 - R1 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & -1 & -1 & 16 \\ 0 & 2 & 0 & -11 \end{array}\right] \quad R3 + 2R2 \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & -1 & -1 & 16 \\ 0 & 0 & -2 & 21 \end{array}\right] \begin{array}{l} R2/(-1) \rightarrow \\ R3/(-2) \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & 1 & 1 & -16 \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] \begin{array}{l} R1 - 2R3 \rightarrow \\ R2 - R3 \rightarrow \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 29 \\ 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] \; R1 + 2R2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -\frac{11}{2} \\ -\frac{21}{2} \end{bmatrix}$$

3. Risolvere il seguente sistema lineare con l'algoritmo di Gauss-Jordan:

$$\begin{cases} 2x_1 + x_2 - x_3 - 3x_4 = -5 \\ 4x_1 + 2x_2 + x_3 - 3x_4 = -7 \\ 2x_1 - x_2 + 3x_3 + 5x_4 = 11 \\ 3x_1 - x_2 - x_3 + x_4 = 8 \end{cases}.$$

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & -5 \\ 4 & 2 & 1 & -3 & -7 \\ 2 & -1 & 3 & 5 & 11 \\ 3 & -1 & -1 & 1 & 8 \end{array}\right] \begin{array}{l} R2 - 2R1 \rightarrow \\ R3 - R1 \rightarrow \\ R4 - \frac{3}{2}R1 \rightarrow \end{array} \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & -5 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & -2 & 4 & 8 & 16 \\ 0 & -\frac{5}{2} & \frac{1}{2} & \frac{11}{2} & \frac{31}{2} \end{array}\right]$$

$$(R2 \leftrightarrow R3) \rightarrow \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & -5 \\ 0 & -2 & 4 & 8 & 16 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & -\frac{5}{2} & \frac{1}{2} & \frac{11}{2} & \frac{31}{2} \end{array}\right] \quad R4 - \frac{5}{4}R2 \rightarrow \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & -5 \\ 0 & -2 & 4 & 8 & 16 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{2} & -\frac{9}{2} \end{array}\right]$$

$$R4 + \frac{3}{2}R3 \rightarrow \left[\begin{array}{cccc|c} 2 & 1 & -1 & -3 & -5 \\ 0 & -2 & 4 & 8 & 16 \\ 0 & 0 & 3 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1/2 \rightarrow \\ R2/(-2) \rightarrow \\ R3/3 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} & -\frac{5}{2} \\ 0 & 1 & -2 & -4 & -8 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{l} R1 + \frac{1}{2}R3 \rightarrow \\ R2 + 2R3 \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & -1 & -2 \\ 0 & 1 & 0 & -2 & -6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \; R1 - \frac{1}{2}R2 \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 & -6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

4. Si usi l'algoritmo di Gauss-Jordan per stabilire se le seguenti matrici sono invertibili e, in caso affermativo, per calcolarne l'inversa:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -5 \\ 1 & -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -5 \\ 1 & 0 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 3 & -5 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2+2R1 \to \\ R3-R1 \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] R3+R2 \to$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \mathbf{A} \text{ non è invertibile}$$

B è la matrice dei coefficienti del sistema lineare di esercizio 2.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 3 & -5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R2+2R1 \to \\ R3-R1 \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] R3+2R2 \to$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 3 & 2 & 1 \end{array}\right] \begin{array}{l} \\ R2/(-1) \to \\ R3/(-2) \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R1-2R3 \to \\ R2-R3 \to \\ \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 4 & 2 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right] R1+2R2 \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 3 & 2 & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{2} & -1 & -\frac{1}{2} \end{bmatrix}$$

5. Siano $a, b, c, d \in \mathbb{R}$ tali che $a \neq 0$, $ad - bc \neq 0$ e sia $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   (a) Si usi l'algoritmo di Gauss-Jordan per calcolare $\mathbf{A}^{-1}$.

   (Osservazione: La formula ottenuta vale anche nel caso $a = 0$, purchè $ad - bc \neq 0$.)

   (b) Si scriva $\mathbf{A}^{-1}$ nel caso $c = d = 0$, cioè l'inversa della matrice $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

$$\left[\begin{array}{rr|rr} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] R2 - \tfrac{c}{a}R1 \to \left[\begin{array}{rr|rr} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \begin{array}{l} R1/a \\ aR2/(ad-bc) \to \end{array}$$

$$\left[\begin{array}{rr|rr} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] R1 - \tfrac{b}{a}R2 \to \left[\begin{array}{rr|rr} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow$$

$$\mathbf{A}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \text{ in particolare nel caso (b): } \begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}.$$