C.d.L. in Scienze naturali – Matematica 09. 01. 2017

1. Dati la matrice $\mathbf{A} = \begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}$ e il vettore $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, calcolare $\mathbf{v}^T\mathbf{A}\mathbf{v}$, dove $\mathbf{v}^T = [x, y]$ è il cosiddetto trasposto di $\mathbf{v}$.

$$\mathbf{v}^T\mathbf{A}\mathbf{v} = [x,y]\begin{bmatrix} 5 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x,y]\begin{bmatrix} 5x-y \\ -x+3y \end{bmatrix} = 5x^2 - 2xy + 3y^2$$

2. Risolvere il seguente sistema lineare con l'algoritmo di Gauss-Jordan:

$$\begin{cases} x_1 - 2x_2 + 2x_3 = 8 \\ -2x_1 + 3x_2 - 5x_3 = 0 \\ x_1 + 2x_3 = -3 \end{cases}.$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ -2 & 3 & -5 & 0 \\ 1 & 0 & 2 & -3 \end{array}\right] \begin{array}{l} \\ R2+2R1 \rightarrow \\ R3-R1 \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & -1 & -1 & 16 \\ 0 & 2 & 0 & -11 \end{array}\right] R3+2R2 \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & -1 & -1 & 16 \\ 0 & 0 & -2 & 21 \end{array}\right] \begin{array}{l} \\ R2/(-1) \rightarrow \\ R3/(-2) \rightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 2 & 8 \\ 0 & 1 & 1 & -16 \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] \begin{array}{l} R1-2R3 \rightarrow \\ R2-R3 \rightarrow \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 29 \\ 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] R1+2R2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & -\frac{11}{2} \\ 0 & 0 & 1 & -\frac{21}{2} \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 18 \\ -\frac{11}{2} \\ -\frac{21}{2} \end{bmatrix}$$

3. Risolvere il seguente sistema lineare con l'algoritmo di Gauss-Jordan:

$$\begin{cases} -x_1 + 3x_2 - 2x_3 = -1 \\ 3x_1 - 8x_2 + 9x_3 = -8 \\ 2x_1 - 5x_2 + 7x_3 = -9 \end{cases}.$$

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ 3 & -8 & 9 & -8 \\ 2 & -5 & 7 & -9 \end{array}\right] \begin{array}{l} \\ R2+3R1 \rightarrow \\ R3+2R1 \rightarrow \end{array} \left[\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ 0 & 1 & 3 & -11 \\ 0 & 1 & 3 & -11 \end{array}\right] R3-R2 \rightarrow$$

$$\left[\begin{array}{ccc|c} -1 & 3 & -2 & -1 \\ 0 & 1 & 3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] R1/(-1) \rightarrow \left[\begin{array}{ccc|c} 1 & -3 & 2 & 1 \\ 0 & 1 & 3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right] R1+3R2 \rightarrow$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 11 & -32 \\ 0 & 1 & 3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right], \text{ cioè } \begin{cases} x_1 + 11x_3 = -32 \\ x_2 + 3x_3 = -11 \\ x_3 = t \text{ (variabile libera)} \end{cases}$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -32 \\ -11 \\ 0 \end{bmatrix} + t\begin{bmatrix} -11 \\ -3 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

4. Risolvere il seguente sistema lineare con l'algoritmo di Gauss-Jordan:

$$\begin{cases} -x_1 + 3x_2 - 2x_3 + 5x_4 = -1 \\ 3x_1 - 8x_2 + 9x_3 - 7x_4 = 5 \\ -2x_1 + 8x_2 - 3x_3 + 6x_4 = -3 \\ 3x_2 + 4x_3 + 4x_4 = 1 \end{cases}.$$

$$\left[\begin{array}{rrrr|r} -1 & 3 & -2 & 5 & -1 \\ 3 & -8 & 9 & -7 & 5 \\ -2 & 8 & -3 & 6 & -3 \\ 0 & 3 & 4 & 4 & 1 \end{array}\right] \begin{array}{l} R2+3R1 \to \\ R3-2R1 \to \end{array} \left[\begin{array}{rrrr|r} -1 & 3 & -2 & 5 & -1 \\ 0 & 1 & 3 & 8 & 2 \\ 0 & 2 & 1 & -4 & -1 \\ 0 & 3 & 4 & 4 & 1 \end{array}\right] \begin{array}{l} R3-2R2 \to \\ R4-3R2 \to \end{array}$$

$$\left[\begin{array}{rrrr|r} -1 & 3 & -2 & 5 & -1 \\ 0 & 1 & 3 & 8 & 2 \\ 0 & 0 & -5 & -20 & -5 \\ 0 & 0 & -5 & -20 & -5 \end{array}\right] R4-R3 \to \left[\begin{array}{rrrr|r} -1 & 3 & -2 & 5 & -1 \\ 0 & 1 & 3 & 8 & 2 \\ 0 & 0 & -5 & -20 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1/(-1) \to \\ \\ R3/(-5) \to \end{array}$$

$$\left[\begin{array}{rrrr|r} 1 & -3 & 2 & -5 & 1 \\ 0 & 1 & 3 & 8 & 2 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R1-2R3 \to \\ R2-3R3 \to \end{array} \left[\begin{array}{rrrr|r} 1 & -3 & 0 & -13 & -1 \\ 0 & 1 & 0 & -4 & -1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] R1+3R2 \to$$

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & -25 & -4 \\ 0 & 1 & 0 & -4 & -1 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 25 \\ 4 \\ -4 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}$$

5. Si usi l'algoritmo di Gauss-Jordan per stabilire se le seguenti matrici sono invertibili e, in caso affermativo, per calcolarne l'inversa:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -5 \\ 1 & -1 & 3 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 3 & -5 \\ 1 & 0 & 2 \end{bmatrix}.$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 3 & -5 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R2+2R1 \to \\ R3-R1 \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] R3+R2 \to$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right] \Rightarrow \mathbf{A} \text{ non è invertibile}$$

B è la matrice dei coefficienti del sistema lineare di esercizio 2.

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ -2 & 3 & -5 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R2+2R1 \to \\ R3-R1 \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 \end{array}\right] R3+2R2 \to$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 3 & 2 & 1 \end{array}\right] \begin{array}{l} R2/(-1) \to \\ R3/(-2) \to \end{array} \left[\begin{array}{rrr|rrr} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & -1 & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right] \begin{array}{l} R1-2R3 \to \\ R2-R3 \to \end{array}$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 0 & 4 & 2 & 1 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right] R1+2R2 \to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3 & 2 & 2 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & -1 & -\frac{1}{2} \end{array}\right]$$

$$\Rightarrow \mathbf{B}^{-1} = \begin{bmatrix} 3 & 2 & 2 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{2} & -1 & -\frac{1}{2} \end{bmatrix}$$

6. Siano $a, b, c, d \in \mathbb{R}$ tali che $a \neq 0$, $ad - bc \neq 0$ e sia $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) Si usi l'algoritmo di Gauss-Jordan per calcolare $\mathbf{A}^{-1}$.

(Osservazione: La formula ottenuta vale anche nel caso $a = 0$, purchè $ad - bc \neq 0$.)

(b) Si scriva $\mathbf{A}^{-1}$ nel caso $c = d = 0$, cioè l'inversa della matrice $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad R2 - \tfrac{c}{a}R1 \rightarrow \quad \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & d - \frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right] \quad \begin{array}{l} R1/a \\ aR2/(ad-bc) \rightarrow \end{array}$$

$$\left[\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \quad R1 - \tfrac{b}{a}R2 \rightarrow \quad \left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right] \Rightarrow$$

$\mathbf{A}^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, in particolare nel caso (b): $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{d} \end{bmatrix}$.