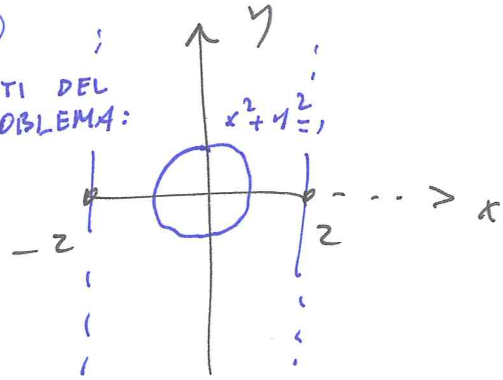
AMTA 10/9/2013

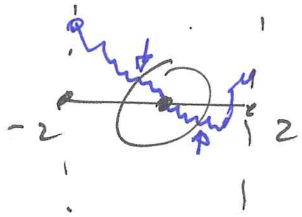
① DATI DEL PROBLEMA:



Chiaramente, $\boxed{\text{se } f(0)=0}$ allora il sistema

$$(*) \begin{cases} y = f(x) \\ x^2+y^2=1 \end{cases}$$

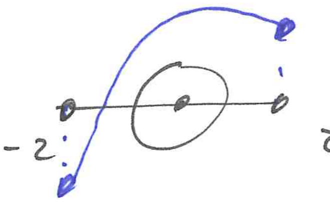
ha almeno una soluzione



(anzi: almeno due!)

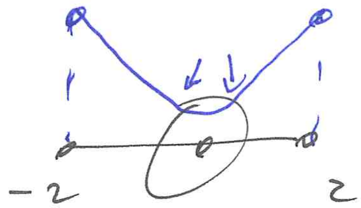
Le altre ipotesi non bastano:

$\boxed{f(-2)<0<f(2)}$ ma

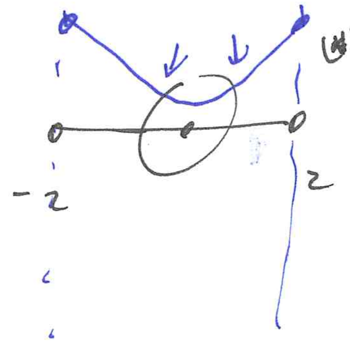


no soluzioni

$\boxed{f(2)>1 \text{ e } f(-2)>1}$ ma



soluzioni



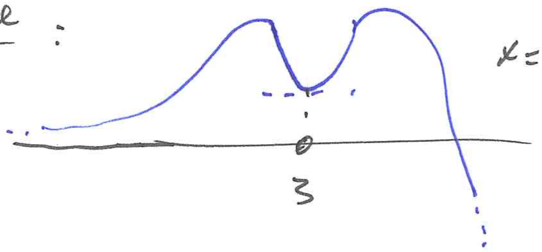
(*) ha soluzioni, ma $f(2)>1$ e $f(-2)>1$

② (a) è vera per T. Lagrange: $\exists c: 2 f'(c) = f(2)-f(0) \overset{\text{ipotesi}}{=} -f(2)$

(b) è vera: $\lim_{n\to\infty} f(-n^2) = \lim_{x\to-\infty} f(x)$ (che esiste per ipotesi)

(d) è vera perché $\lim_{n\to\infty} f\left(-\frac{1}{n+1}\right) = \lim_{x\to 0} f(x) = f(0)$

(c) è falsa:



$x=3$ è min. rel., non minimo.

① $f(x) = e^{h(11+13x^2)} \cdot \log\left(1+\cos^2(13^2x^4+11)\right)$

$$f'(x) = e^{h(11+13x^2)} \cdot h'(11+13x^2)\cdot 13\cdot 2x + e^{h(11+13x^2)} \cdot \frac{2\cdot\cos(13^2x^4+11)\cdot[-\sin(13^2x^4+11)]\cdot 13^2\cdot 4x}{1+\cos^2(13^2x^4+11)}$$

$$f'\left(\frac{1}{\sqrt{13}}\right) = e^{h(12)}\cdot h'(12)\cdot 2\sqrt{13} + e^{h(12)}\cdot \frac{2\cdot\cos(12)\,[-\sin(12)]\cdot\sqrt{13}\cdot 4}{1+\cos^2(12)}$$

$$= e^{13}\cdot 26 - e^{13}\cdot 8\cdot\sqrt{13}\cdot \frac{\cos(12)\sin(12)}{1+\cos^2(12)}$$

② $$\left(1+\frac{1}{n^2}\right)^n + \left(1+\frac{1}{n^2}\right)^{n^2} + \left(1-\frac{1}{n}\right)^{n^2} =$$

$$= \left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{1/n} + \left(1+\frac{1}{n^2}\right)^{n^2} + \left[\left(1+\frac{-1}{n}\right)^{-n}\right]^{-n}$$

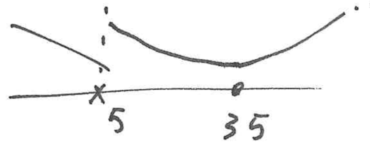
$$\xrightarrow[n\to\infty]{} e^0 + e + e^{-\infty} = 1+e$$

③ Dominio$(f) = \mathbb{R}\setminus\{5\}$ e $f \in C(\mathbb{R}\setminus\{5\})$.

Di più, $f \in C^1(\mathbb{R}\setminus\{5\})$ e

$$\forall x \neq 5:\ f'(x) = \frac{(x-5)-(x+25)}{(x-5)^2} + \frac{1}{x-5} = \frac{-30+(x-5)}{(x-5)^2} = \frac{x-35}{(x-5)^2}$$

Avendo $f'(x) \geq 0 \iff x-35 \geq 0 \iff x \geq 35$ abbiamo che f decresce su $(-\infty, 5)$ e su $(5, 35]$, mentre cresce su $[35, +\infty)$.



$x=35$ è p.to di minimo relativo.

Calcolo: $\lim_{x\to\pm\infty} f(x) = 1+\log(+\infty) = +\infty$

$\lim_{x\to 5^+} f(x) = +\infty-\infty$: vediamo meglio. Poiché $\lim_{x\to 5^+} (x-5)\log|x-5| = 0$,

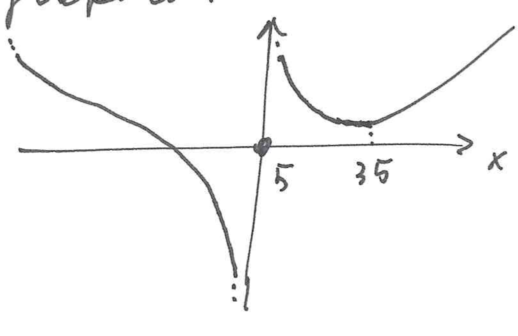
quindi $f(x) =$

$$= \frac{(x+25)+(x-5)\log|x-5|}{x-5} \xrightarrow[x\to 5^+]{} \frac{30+0}{0^+} = +\infty$$

cioè: $\lim_{x\to 5^+} f(x) = +\infty$

$\lim_{x\to 5^-} f(x) = -\infty-\infty = -\infty$

Grafico:



(4) $$e^{e^{3x}-1} = e^{1+3x+(3x)^2/2+(3x)^3/6+o(x^3)-1}$$

$$= e^{3x+9/2\,x^2+9/2\,x^3+o(x^3)}$$

$$= 1+\left[3x+9/2\,x^2+9/2\,x^3+o(x^3)\right]+\frac{1}{2}\left[3x+9/2\,x^2+o(x^2)\right]^2$$

$$+\frac{1}{6}\left[3x+o(x)\right]^3+o(x^3)$$

$$= 1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\frac{9}{2}x^2+\frac{1}{2}\cdot 2\cdot 3\cdot\frac{9}{2}x^3+\frac{3^2}{2}x^3+o(x^3)$$

$$= 1+3x+9x^2+\left(\frac{9}{2}+\frac{27}{2}+\frac{9}{2}\right)x^3+o(x^3)$$

$$= 1+3x+9x^2+\frac{5\cdot 9}{2}x^3+o(x^3)$$

Quindi $e^{e^{3x}-1}-\frac{1}{1-3x}=$

$$=\left[1+3x+9x^2+\frac{5\cdot 9}{2}x^3+o(x^3)\right]-\left[1+3x+9x^2+27x^3+o(x^3)\right]$$

$$=\left[\frac{5\cdot 9}{2}-27\right]x^3+o(x^3)=-\frac{9}{2}x^3+o(x^3),$$

quindi $\lim_{x\to 0}\left[e^{e^{3x}-1}-\frac{1}{1-3x}\right]/\left[7x^3+\sin(x^4)\right]$

$$=\lim_{x\to 0}\frac{-\frac{9}{2}x^3+o(x^3)}{x^3\cdot\left[7+\frac{\sin(x^4)}{x^4}\cdot x\right]}=\frac{-9/2}{7}=-\frac{9}{14}$$

(5) $$\int_{-3}^{3}\cos(2x)\cdot\left[\sin(2x)+3\right]^2\cdot e^{3\sin(2x)+6}\,dx \overset{\sin(2x)+3=y}{=}$$

$$=\int_{\sin(-6)+3=\alpha}^{\sin(6)+3=\beta} y^2 e^{3y-3}\,\frac{dy}{2}=\frac{e^{-3}}{2}\left\{\left[\frac{e^{3y}}{3}y^2\right]_\alpha^\beta-\int_\alpha^\beta\frac{e^{3y}}{3}2y\,dy\right\}$$

$$=\frac{e^{-3}}{2}\cdot\left\{\left[\frac{e^{3y}}{3}y^2\right]_\alpha^\beta-\left[\frac{e^{3y}}{3^2}2y\right]_\alpha^\beta+\int_\alpha^\beta\frac{2}{3^2}e^{3y}\,dy\right\}$$

$$=\frac{e^{-3}}{2}\cdot\left\{\left[\frac{e^{3y}}{3}y^2\right]_\alpha^\beta-\left[\frac{e^{3y}}{3^2}2y\right]_\alpha^\beta+\left[\frac{e^{3y}}{3^3}2\right]_\alpha^\beta\right\}.$$