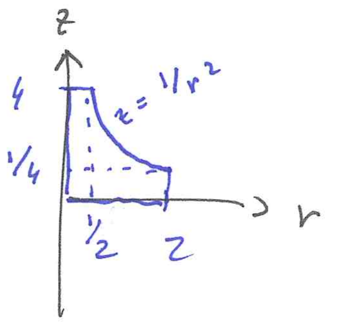
AM II 10/9/2013

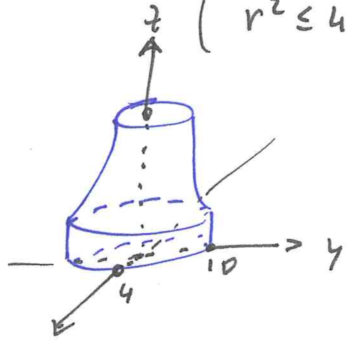
(1)

(1) $\Omega = \{(x,y,z):\ 0 \le \left(\frac{x^2}{4}+\frac{y^2}{25}\right)\cdot z \le 1;\ z\le 4;\ \frac{x^2}{4}+\frac{y^2}{25}\le 4\}$.

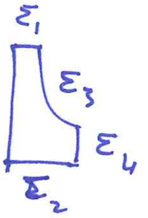
(101) Posti $\begin{cases} \frac{x}{2} = r\cos\theta \\ \frac{y}{5} = r\sin\theta\end{cases}$ con $r\ge 0,\ |\theta|\le\pi$, ho che

$$(x,y,z)\in\Omega \iff \begin{cases} |\theta|\le\pi \\ 0\le r^2\cdot z\le 1 \\ z\le 4 \\ r^2\le 4\end{cases} \iff \begin{cases}|\theta|\le\pi\\ 0\le r\le 2 \\ 0\le z\le 4 \\ z\le 1/r^2\end{cases} \quad (*)$$





(102)



$\Sigma_1 = \{(x,y,4):\ \frac{x^2}{4}+\frac{y^2}{25}\le\frac{1}{4}\} = \Phi_1(A_1)$ dove:

$\mathbb{R}^2 \supseteq A_1 = \{(x,y):\ \frac{x^2}{4}+\frac{y^2}{25}\le\frac{1}{4}\} \xrightarrow{\Phi_1} \mathbb{R}^3 \quad \Phi_1(x,y) = (x,y,4)$;

$\partial_x\Phi_1 \times \partial_y\Phi_1(x,y) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{vmatrix} = (0,0,1)$: compatibile con ν

$\Sigma_2 = \{(x,y,0):\ \frac{x^2}{4}+\frac{y^2}{25}\le 4\} = \Phi_2(A_2)$ dove $A_2 = \{(x,y):\ \frac{x^2}{4}+\frac{y^2}{25}\le 4\} \xrightarrow{\Phi_2} \mathbb{R}^3$

e $\partial_x\Phi_2 \times \partial_y\Phi_2(x,y) = (0,0,1)$: non compatibile con ν.

$\Sigma_3 = \{(x,y,z):\ \left(\frac{x^2}{4}+\frac{y^2}{25}\right) z = 1;\ \frac{1}{4}\le z\le 4\} = \Phi_3(A_3)$ dove

$\Phi_3(r,\theta) = \left(2r\cos\theta,\ 5r\sin\theta,\ \frac{1}{r^2}\right)$; $A_3 = [\frac{1}{2}, 2]\times[-\pi,\pi] \xrightarrow{\Phi_3} \mathbb{R}^3$

e $\partial_r\Phi_3 \times \partial_\theta\Phi_3(r,\theta) = \begin{vmatrix} i & j & k \\ 2\cos\theta & 5\sin\theta & -\frac{2}{r^3} \\ -2r\sin\theta & 5r\cos\theta & 0\end{vmatrix} = \left(\frac{10}{r^2}\cos\theta,\ \frac{4}{r^2}\sin\theta,\ 10r\right)$ compatibile con ν

$\Sigma_4 = \{(x,y,z):\ \frac{x^2}{4}+\frac{y^2}{25} = 4;\ 0\le z\le \frac{1}{4}\} = \Phi_4(A_4)$

$\Phi_4(\theta,z) = (4\cos\theta,\ 10\sin\theta,\ z)$; $\Phi_4: A_4 = [-\pi,\pi]\times[0,\frac{1}{4}] \to \mathbb{R}^3$

e $\partial_\theta\Phi_4 \times \partial_z\Phi_4 = \begin{vmatrix} i & j & k \\ -4\sin\theta & 10\cos\theta & 0 \\ 0 & 0 & 1\end{vmatrix} = (10\cos\theta,\ 4\sin\theta,\ 0) \underset{\theta=0}{=} (10,0,0)$

compatibile con ν.

(10.3) $\displaystyle\iint_{\partial\Omega} F\cdot\nu\,d\sigma = \iiint_{\Omega} \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \iiint_{\{(\theta,r,z)\text{ come in }(*)\}} \operatorname{div} F(2r\cos\theta, 5r\sin\theta, z)\,10r\,dr\,d\theta\,dz$

$$= \int_{-\pi}^{\pi} d\theta \int_0^{1/2} 10r\,dr \int_0^4 dz\cdot \operatorname{div} F(2r\cos\theta, 5r\sin\theta, z)$$

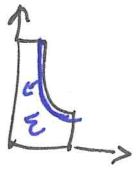
$$+ \int_{-\pi}^{\pi} d\theta \int_{1/2}^{2} 10r\,dr \int_0^{1/r^2} dz\cdot \operatorname{div} F(2r\cos\theta, 5r\sin\theta, z)$$

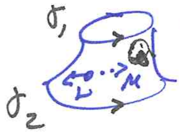
(10.4) Avendo per $F(x,y,z) = (y,y,0)$ che $\operatorname{div} F(x,y,z) = 1$,

$$\iint_{\partial\Omega} F\cdot\nu\,d\sigma = 2\pi\cdot 10\cdot\left(\frac{r^2}{2}\right)\Big|_0^{1/2}\cdot 4 + 2\pi\cdot\int_{1/2}^{2} 10\cdot r\cdot\frac{1}{r^2}\,dr$$

$$= 10\cdot\pi\cdot\frac14 + 2\pi\cdot 10\cdot\left(\log 2 - \log\frac12\right) = \boxed{5\pi\cdot\left(\frac12 + 8\log 2\right)}$$

(10.5)





$\Sigma = \Sigma_3$

$\gamma_1(\theta) = ($ [illegible] $\cos\theta, \frac52\sin\theta, 4)$ comp.

$\gamma_2(\theta) = (4\cos\theta, 10\sin\theta, 1/4)$ non comp.

$\gamma_1, \gamma_2 : [-\pi,\pi] \to \mathbb{R}^3$

(10.6)

$$\iint_{\Sigma} (\nabla\times F)\cdot\mu\,d\sigma = \int_{\partial\Sigma} F(\gamma)\cdot d\gamma =$$

$$= \int_{-\pi}^{\pi} \left(\frac52\sin\theta, \frac52\sin\theta, 0\right)\cdot\left(-\sin\theta, \frac52\cos\theta, 0\right)d\theta$$

$$- \int_{-\pi}^{\pi} (10\sin\theta, 10\sin\theta, 0)\cdot(-4\sin\theta, 10\cos\theta, 0)\,d\theta$$

$$= \int_{-\pi}^{\pi}\left[\left(40-\frac52\right)\sin^2\theta + \left(\frac{25}{4}-100\right)\cos\theta\sin\theta\right]d\theta$$

$$= \left(40-\frac52\right)\cdot\pi + \left(\frac{25}{4}-100\right)\cdot 0 = \frac{75}{2}\pi$$

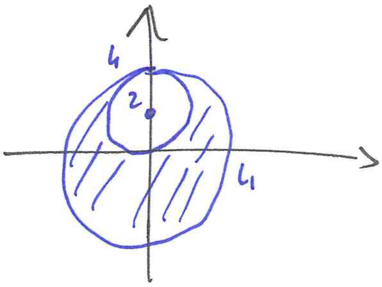
(2) $F = (P,Q)$: $\displaystyle P_y(x,y) = \frac{-x\cdot 4\alpha y\cdot(x^2+y^2)}{[1+\alpha(x^2+y^2)^2]^2} \overset{?}{=} Q_x(x,y) = \frac{-y\cdot(1-3\alpha)\cdot 4x\cdot(x^2+y^2)}{[1+(1-3\alpha)(x^2+y^2)^2]^2}$

$\Updownarrow$

$\alpha = 1-3\alpha \iff \alpha = 1/4$

F è chiuso (quindi esatto: è definito su $\mathbb{R}^2$) $\iff \alpha = 1/4$

(3)



$$\iint_A dx\,dy = \text{Area}(A) = \pi\cdot 4^2 - \pi\cdot 2^2 = \pi\cdot(16-4)$$

$$= 12\cdot\pi$$

(4) $\frac{d}{dt}(\dot{x}+3x)=t \iff \dot{x}+3x=\frac{t^2}{2}+A \qquad (A\in\mathbb{R})$

Provo con $x(t)=\cancel{at^3}+bt^2+ct+d$

$\dot{x}(t)=\cancel{3at^2}+2bt+c$

$$\frac{t^2}{2}+A=\dot{x}+x=t^2(3b)+t(3c+2b)+(3d+c)$$

$$\iff \begin{cases} 3b=1/2 \\ 3c+2b=0 \\ 3d+c=A \end{cases} \iff \begin{cases} b=1/6 \\ c=-\frac{2}{3}b=-\frac{1}{9} \\ d=\frac{A-c}{3}=\frac{A}{3}+\frac{1}{27} \end{cases}$$

Omogenea: $\dot{z}+3z=0 \qquad \lambda+3=0: \; z(t)=e^{-3t}$

Int. gen.: $x(t)=\frac{t^2}{6}-\frac{t}{9}+\frac{A}{3}+\frac{1}{27}+Be^{-3t}$

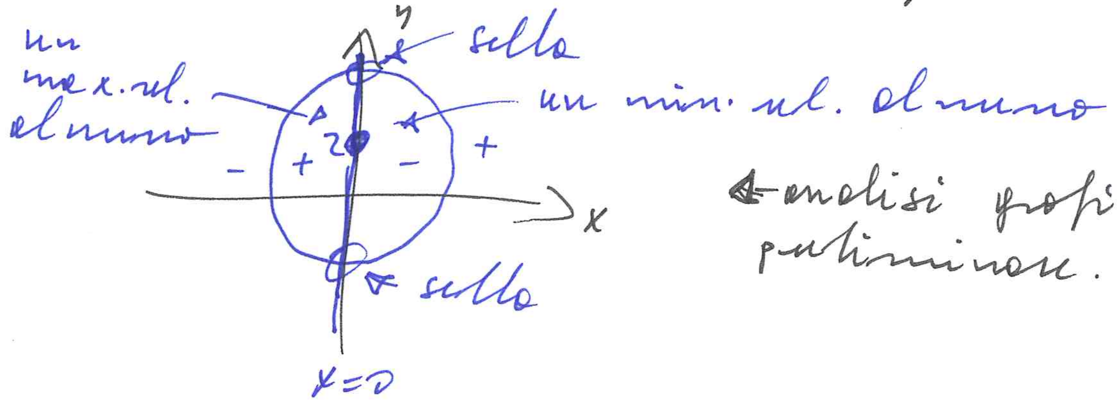
$$=\frac{t^2}{6}-\frac{t}{9}+C_1+Be^{-3t}$$

(5) Posto $h=h(u,v)$:

$$\partial_x f(x,y,z)=\partial_u h(2\alpha(x,y,z),5\beta(x,y,z))\cdot 2\cdot\partial_x\alpha(x,y,z)$$
$$+\partial_v h(2\alpha(x,y,z),5\beta(x,y,z))\cdot 5\,\partial_x\beta(x,y,z)$$

$$\partial_y f(x,y,z)=\partial_u h(2\alpha(x,y,z),5\beta(x,y,z))\cdot 2\,\partial_y\alpha(x,y,z)$$
$$+\partial_v h(2\alpha(x,y,z),5\beta(x,y,z))\cdot 5\,\partial_y\beta(x,y,z)$$

(6)



analisi grafica preliminare.

$$\begin{cases} f_x(x,y)=[(y-2)^2+x^2-16]+2x\cdot x \\ f_y(x,y)=2(y-2)\cdot x \end{cases} \Rightarrow \nabla f(x,y)=0 \iff$$

$$\underbrace{\begin{cases} x=0 \\ (y-2)^2+x^2-16=0 \end{cases}}_{\text{selle}} \text{ o } \underbrace{\begin{cases} y=2 \\ x^2-16+2x^2=0 \end{cases}}_{\text{MAX/min rel}}$$

$$\underbrace{\begin{cases} x=0 \\ y=2\pm4 \end{cases}}_{\text{selle}} \text{ o } \underbrace{\begin{cases} y=2 \\ x=\pm\frac{4}{\sqrt{3}} \end{cases}}_{\text{MAX/min rel.}}$$

$\boxed{\begin{matrix}(0,6)\\(0,-2)\end{matrix}}$ selle

$\left(\frac{4}{\sqrt{3}},2\right) \to$ min. rel.

$\left(-\frac{4}{\sqrt{3}},2\right) \to$ MAX. rel.