

Hardy's inequalities for monotone functions on partially ordered measure spaces

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Abstract. We characterize the weighted Hardy's inequalities for monotone functions in \mathbb{R}_+^n . In dimension $n = 1$, this recovers the classical theory of B_p weights. For $n > 1$, the result was only known for the case $p = 1$. In fact, our main theorem is proved in the more general setting of partially ordered measure spaces.

1 Introduction

The theory of weighted inequalities for the Hardy operator, acting on monotone functions in \mathbb{R}_+ , was first introduced in [2]. Extensions of these results to higher dimension have been considered only in very specific cases. In particular, in the diagonal case, only for $p = 1$ (see [5]). The main difficulty in this context is that the level sets of the monotone functions are not totally ordered, contrary to the one-dimensional case where one considers intervals of the form $(0, a)$, $a > 0$. It is also worth to point out that, with no monotonicity restriction, the boundedness of the Hardy operator is only known in dimension $n = 2$ (see [15], [12], and also [3] for an extension in the case of product weights).

In this work we completely characterize the weighted Hardy's inequalities for all values of $p > 0$, namely, the boundedness of the operator:

$$S : L_{\text{dec}}^p(u) \longrightarrow L^p(u),$$

where

$$Sf(s, t) = \frac{1}{st} \int_0^s \int_0^t f(x, y) dy dx,$$

and $L_{\text{dec}}^p(u)$ is the cone of positive and decreasing functions, on each variable, in $L^p(u) = L^p(\mathbb{R}_+^2, u(x) dx)$ (we consider, for simplicity, $n = 2$, although the result holds in any dimension).

The techniques we are going to use were introduced in [8], for the one-dimensional case, and apply also to a more general setting, which we now define:

We will consider a family of σ -finite measure spaces (X, μ_x) (where μ_x is a measure on X , for each $x \in X$), with a partial order \leq satisfying:

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1. If $X_x := \{u \in X : u \leq x\}$, then \leq restricted to X_x is a total order.
2. If D is a decreasing set, with respect to the order \leq (i.e., χ_D is a decreasing function), then D is measurable.
3. $\mu_x(X_x) = 1$ (observe that X_x is a decreasing set).
4. If $u \in X_x$, then $d\mu_x(y) = \mu_x(X_u)d\mu_u(y)$. In particular, $\mu_x(X_u)\mu_u(X_x) = 1$.

The main examples are:

- $X = \mathbb{R}_+$ with the usual order, and $\mu_x(E) = x^{-1}|E|$. This is the case considered in [2].
- X is a tree with the usual order on geodesics, and $\mu_x(E) = \text{Card}(E)/|x|$, where $|x| = \text{Card}([o, x])$ and $[o, x]$ is the geodesic path joining the origin o with any vertex x of the tree. For more information on this case, see [10] and the references quoted therein.
- \mathbb{R}_+^2 with the order given by $(a_1, b_1) \leq (a_2, b_2)$ if and only if, $a_1 = a_2$ and $b_1 \leq b_2$ (we could also choose to fix the second coordinate). For $x = (a, b) \in \mathbb{R}_+^2$,

$$\mu_x(E) = b^{-1} \int_{E \cap (\{a\} \times \mathbb{R}_+)} dt.$$

- In many cases, we can easily get the existence of the family of measures μ_x by taking a non-negative measure μ on X , and defining $\mu_x = \mu/\mu(X_x)$.

We now define the Hardy operator as follows:

$$Sf(x) = \int_{X_x} f(u) d\mu_x(u).$$

This definition is similar to the one considered in [5]. For the case of \mathbb{R}_+ ,

$$Sf(x) = \frac{1}{x} \int_0^x f(t) dt.$$

On a tree,

$$Sf(x) = \sum_{y \in [o, x]} \frac{f(y)}{|x|},$$

and for \mathbb{R}_+^2 ,

$$Sf(a, b) = \frac{1}{b} \int_0^b f(a, t) dt.$$

One of the main techniques we are going to use is the following lemma. This is a kind of integration by parts:

Lemma 1.1 *Let (X, μ, \leq) be a finite measure space with a total order \leq , and $\alpha \in \mathbb{R}$. Then, there exists a constant C_α which depends only on α (and not on (X, μ, \leq)), such that*

$$\left(\int_X d\mu \right)^\alpha \leq C_\alpha \int_X \left(\int_{X_u} d\mu \right)^{\alpha-1} d\mu(u).$$

Proof: Since $\mu(X) < \infty$, dividing both sides by $(\mu(X))^\alpha$, it suffices to show that

$$1 \leq C_\alpha \int_X \varphi^{\alpha-1}(u) d\mu(u),$$

where $\varphi(u) = \mu(X_u)$, and $\mu(X) = 1$.

If $\alpha \leq 1$, using that $0 \leq \varphi \leq 1$, then $\varphi^{\alpha-1}(u) \geq 1$, and

$$\int_X \varphi^{\alpha-1}(u) d\mu(u) \geq \mu(X) = 1.$$

If $1 < \alpha \leq 2$, then $\varphi^{\alpha-1}(u) \geq \varphi(u)$, and hence it suffices to prove it for $\alpha = 2$.

If $2 < \alpha$, using Jensen's inequality:

$$\left(\int_X \varphi(u) d\mu(u) \right)^{\alpha-1} \leq \int_X \varphi^{\alpha-1}(u) d\mu(u),$$

and as before, it reduces to the case $\alpha = 2$.

Finally, if $\alpha = 2$,

$$\begin{aligned} \int_X \left(\int_{X_u} d\mu(x) \right) d\mu(u) &= \int_X \left(\int_{\{x \leq u\}} d\mu(u) \right) d\mu(x) \\ &= \int_X \left(1 - \int_{\{x > u\}} d\mu(u) \right) d\mu(x) \\ &= 1 - \int_X \int_{\{u \leq x\}} d\mu(u) d\mu(x) + \int_X \int_{\{u=x\}} d\mu(u) d\mu(x), \end{aligned}$$

(here we used that the order is total). Thus,

$$\int_X \left(\int_{X_u} d\mu(x) \right) d\mu(u) \geq \frac{1}{2}.$$

□

2 Weighted Hardy's inequality

In this section we will prove the main theorem. In order to include all the examples, we need to consider a second weaker order \prec satisfying:

1. If $x \leq y$, then $x \prec y$.
2. If f is \prec -decreasing, then Sf is \prec -decreasing.

We recall that we still keep \leq to define the operator S .

Remark 2.1

- We can (and will in some cases) take \prec to be \leq . In fact, we only need to check that the second condition holds for \leq : If f is \leq -decreasing, and $u \leq x$, then $Sf(x) \leq Sf(u)$ if and only if,

$$\int_{X_u} f(y)[1 - \mu_x(X_u)] d\mu_u(y) - \int_{X_x \setminus X_u} f(y)\mu_x(X_u) d\mu_u(y) \geq 0,$$

(here we have used that $d\mu_x(y) = \mu_x(X_u)d\mu_u(y)$), and this follows from the fact that $\inf f|_{X_u} \geq \sup f|_{X_x \setminus X_u}$.

- If a function f or a set D are \prec -decreasing they are also \leq -decreasing.
- The main example we have in mind is \leq in \mathbb{R}_+^2 as before, and \prec the order given by the rectangles (which is clearly a weaker order): $(a_1, b_1) \prec (a_2, b_2)$, if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$ (i.e., the rectangle in \mathbb{R}_+^2 determined by the origin and (a_1, b_1) is contained in the one determined by the origin and (a_2, b_2)). To show the second condition, assume that f is a function decreasing on each variable. Then, it is obvious that $y^{-1} \int_0^y f(x, t) dt$ is also decreasing on each variable.
- We denote by $L_{\prec}^p(d\nu)$ the class of \prec -decreasing functions in $L^p(d\nu)$. As a general assumption, we will only consider cases for which measurability of the functions involved always holds.

Theorem 2.2 *Let (X, μ_x, \leq) and \prec satisfy the conditions given above, for all $x \in X$. Let $d\nu$ be a measure on X , and $p > 0$. Then, the Hardy operator is bounded*

$$S : L_{\prec}^p(d\nu) \rightarrow L^p(d\nu),$$

if and only if, there exist a constant $C > 0$ such that, for all \prec -decreasing sets D ,

$$\int_{X \setminus D} \mu_x^p(D \cap X_x) d\nu(x) \leq C\nu(D). \tag{1}$$

Proof: Consider $f = \chi_D$, where D is \prec -decreasing. Then $(Sf)^p(x) = \mu_x^p(D \cap X_x)$, and hence

$$\begin{aligned} \|Sf\|_{L^p(d\nu)}^p &= \int_X \mu_x^p(D \cap X_x) d\nu(x) \\ &= \int_D \mu_x^p(D \cap X_x) d\nu(x) + \int_{X \setminus D} \mu_x^p(D \cap X_x) d\nu(x) \\ &\leq C\nu(D). \end{aligned}$$

Thus,

$$\int_{X \setminus D} \mu_x^p(D \cap X_x) d\nu(x) \leq C\nu(D).$$

Observe that since D is also \leq -decreasing, if $x \in D$, then $X_x \subset D$, and so $\mu_x^p(D \cap X_x) = \mu_x^p(X_x) = 1$. Therefore, $\int_D \mu_x^p(D \cap X_x) d\nu(x) = \nu(D)$.

Conversely, if $p \geq 1$ and $f \in L^p_<(d\nu)$, using the lemma with (X_x, μ, \leq) and $d\mu(u) = f(u)d\mu_x(u)$, we have that, for a constant C which does not depend on either f or x ,

$$\begin{aligned} (Sf)^p(x) &= \left(\int_{X_x} f(u) d\mu_x(u) \right)^p \leq C \int_{X_x} \left(\int_{X_u} f(y) d\mu_x(y) \right)^{p-1} f(u) d\mu_x(u) \\ &= C \int_{X_x} \left(\int_{X_u} f(y) d\mu_u(y) \right)^{p-1} f(u) \mu_x^{p-1}(X_u) d\mu_x(u) \\ &= C \int_0^\infty \int_{\{g>t\} \cap X_x} \mu_x^{p-1}(X_u) d\mu_x(u) dt, \end{aligned}$$

where $g(u) = (Sf)^{p-1}(u)f(u)$. Hence,

$$\begin{aligned} \|Sf\|_{L^p(d\nu)}^p &\leq C \int_X \int_0^\infty \int_{\{g>t\} \cap X_x} \mu_x^{p-1}(X_u) d\mu_x(u) dt d\nu(x) \\ &\approx \int_X \int_0^{g(x)} \int_{\{g>t\} \cap X_x} \mu_x^{p-1}(X_u) d\mu_x(u) dt d\nu(x) \\ &\quad + \int_X \int_{g(x)}^\infty \int_{\{g>t\} \cap X_x} \mu_x^{p-1}(X_u) d\mu_x(u) dt d\nu(x) \\ &= I + II. \end{aligned}$$

Since $X_u \subset X_x$, if $u \in X_x$, and $p \geq 1$, then

$$I \leq \int_X \int_0^{g(x)} \mu_x^{p-1}(X_x) \mu_x(X_x) dt d\nu(x) = \int_X g(x) d\nu(x).$$

Since both Sf and f are \prec -decreasing, and $p \geq 1$, then g is \prec -decreasing and $\{g > t\}$ is a \prec -decreasing set. Also, if $u \in \{g > t\} \cap X_x$, then $X_u \subset \{g > t\} \cap X_x$, and hence

$$\int_{\{g>t\} \cap X_x} \mu_x^{p-1}(X_u) d\mu_x(u) \leq \mu_x^p(\{g > t\} \cap X_x).$$

Therefore, using the hypothesis,

$$\begin{aligned} II &\leq \int_0^\infty \int_{X \setminus \{g > t\}} \mu_x^p(\{g > t\} \cap X_x) d\nu(x) dt \\ &\leq C \int_0^\infty \int_{\{g > t\}} d\nu(x) dt = C \int_X g(x) d\nu(x). \end{aligned}$$

So, using Hölder's inequality,

$$\|Sf\|_{L^p(d\nu)}^p \leq C \int_X (Sf)^{p-1}(x) f(x) d\nu(x) \leq C \|Sf\|_{L^p(d\nu)}^{p-1} \|f\|_{L^p(d\nu)}.$$

From this a priori estimate, one obtains the general result by a standard density argument.

If $0 < p < 1$, and $f \in L^p_{\prec}(d\nu)$, set $D_t = \{f > t\}$, and

$$g_x(t) = \int_{D_t \cap X_x} d\mu_x(u).$$

Then, using the embedding $L^p_{\text{dec}}(t^{p-1}) \hookrightarrow L^1$ (see [14]), we have (observe that g_x is a decreasing function):

$$\begin{aligned} \left(\int_X \left(\int_{X_x} f(u) d\mu_x(u) \right)^p d\nu(x) \right)^{1/p} &= \left(\int_X \left(\int_0^\infty g_x(t) dt \right)^p d\nu(x) \right)^{1/p} \\ &\leq C \left(\int_X \int_0^\infty t^{p-1} (g_x(t))^p dt d\nu(x) \right)^{1/p}. \end{aligned}$$

Since $g_x(t) \leq \mu_x(X_x) = 1$, then

$$\int_0^\infty t^{p-1} \int_{D_t} (g_x(t))^p d\nu(x) dt \leq \int_0^\infty t^{p-1} \int_{D_t} d\nu(x) dt = \frac{1}{p} \|f\|_{L^p(d\nu)}^p.$$

On the other hand, using the hypothesis (observe that $D_t \cap X_x$ is a decreasing set),

$$\int_0^\infty t^{p-1} \int_{X \setminus D_t} (g_x(t))^p d\nu(x) dt \leq C \int_0^\infty t^{p-1} \int_{D_t} d\nu(x) dt = \frac{C}{p} \|f\|_{L^p(d\nu)}^p.$$

Therefore,

$$\left(\int_X \left(\int_{X_x} f(u) d\mu_x(u) \right)^p d\nu(x) \right)^{1/p} \leq C \|f\|_{L^p(d\nu)}.$$

□

The following results follow easily by particularizing on each case condition (1) of Theorem 2.2:

Corollary 2.3

1. **(Case of equality of orders \prec and \leq)** Let (X, μ_x, \leq) satisfy the conditions given above. Let $d\nu$ be a measure on X , and $p > 0$. Then, the Hardy operator is bounded

$$S : L^p_{\leq}(d\nu) \rightarrow L^p(d\nu),$$

if and only if, there exist a constant $C > 0$ such that, for all \leq -decreasing sets D ,

$$\int_{X \setminus D} \mu_x^p(D \cap X_x) d\nu(x) \leq C\nu(D).$$

2. **(Case of \mathbb{R}_+)** Condition (1) of Theorem 2.2 is:

$$\int_r^\infty \left(\frac{r}{x}\right)^p d\nu(x) \leq C \int_0^r d\nu(x),$$

for all $r > 0$, which is B_p .

3. **(Case of a tree T)** Condition (1) of Theorem 2.2 is:

$$\sum_{x \in T \setminus D} \frac{|x \vee D|^p}{|x|^p} \nu(x) \leq C \sum_{x \in D} \nu(x),$$

where $x \vee D$ is the largest vertex in $[o, x] \cap D$.

4. **(Case of \mathbb{R}_+^2)** Condition (1) of Theorem 2.2 is:

$$\int_{\mathbb{R}_+^2 \setminus D} \frac{|D_x|^p}{t^p} d\nu(x, t) \leq C \int_D d\nu(x, t),$$

where $D_x = \{t > 0 : (x, t) \in D\}$, and D is any decreasing set (on each variable).

Remark 2.4 As we have mentioned above, the case of \mathbb{R}_+ was first considered in [2]. B_p weights are well understood and enjoy a very rich structure (see also [14], [6], and [7] for an account of B_p and normability properties of weighted Lorentz spaces).

The discrete case \mathbb{N} is a particular case of a tree, and can be found in [7]. Weights for a general tree were studied, without the monotonicity condition, in [9] and [1]. It is easy to prove that a weight satisfying Corollary 2.3 (3) must necessarily be in $B_p(\mathbb{N})$ (uniformly) on each geodesic (see [7]), but the converse is not true in general.

We want to give a new example which shows that, even in dimension one ($X = \mathbb{R}_+$), Theorem 2.2 can provide new results:

Set $X_n = (n, n + 1)$, $n \in \mathbb{N} \cup \{0\}$, and define the order $x \blacktriangleleft y$ if and only if $x, y \in X_n$ (i.e., $[x] = [y]$) and $x \leq y$ (the usual order in \mathbb{R}_+). Define also the measures $d\mu_x(y) = dy/(x - [x])$ (in fact we only consider non-integer positive numbers). We introduce three different kinds of additional orders:

- $\prec_1 = \blacktriangleleft$.
- $x \prec_2 y \Leftrightarrow [x] \leq [y]$ and $x - [x] \leq y - [y]$.
- $x \prec_3 y \Leftrightarrow [x] < [y]$ or $x \blacktriangleleft y$ (i.e., $x \leq y$).

It is easy to check that, in all cases, the orders \blacktriangleleft and \prec_j , $j = 1, 2, 3$, satisfy the hypothesis of Theorem 2.2. Observe that, for this example (i.e., for the order \blacktriangleleft), the operator is of the form

$$S_{\blacktriangleleft}f(x) = \frac{1}{x - [x]} \int_{[x]}^x f(t) dt,$$

which is a Hardy type operator with variable end-points (more general operators, with no monotonicity conditions on the function, have been considered in [11]). For each $j = 1, 2, 3$ we have that the function f is \prec_j -decreasing if:

- *Case \prec_1 :* f restricted to $(n, n + 1)$ is decreasing, for every $n \in \mathbb{N} \cup \{0\}$. The decreasing sets are of the form $\cup_{n \geq 0} (n, n + a_n)$, where $a_n \in (0, 1)$.
- *Case \prec_2 :* f restricted to $(n, n + 1)$ is decreasing, for every $n \in \mathbb{N} \cup \{0\}$, and $f(x) \geq f(x + 1)$, $x > 0$. The decreasing sets are of the form $\cup_{n \geq 0} (n, n + a_n)$, where $0 < a_{n+1} < a_n < 1$.
- *Case \prec_3 :* f is decreasing for the usual order \leq . The decreasing sets are of the form $(0, a)$, $a > 0$.

A direct application of Theorem 2.2 gives:

Corollary 2.5 *Let u be a weight in \mathbb{R}_+ . Then, $S_{\prec_j} : L^p_{\prec_j}(u(x) dx) \rightarrow L^p(u(x) dx)$ ($j = 1, 2, 3$), if and only if,*

Case \prec_1 : For every sequence $\{a_n\}_n \subset [0, 1]$,

$$\sum_{n=0}^{\infty} \int_{a_n}^1 \left(\frac{a_n}{t}\right)^p u(n+t) dt \leq C \sum_{n=0}^{\infty} \int_0^{a_n} u(n+t) dt. \quad (2)$$

Case \prec_2 : For every decreasing sequence $\{a_n\}_n \subset [0, 1]$,

$$\sum_{n=0}^{\infty} \int_{a_n}^1 \left(\frac{a_n}{t}\right)^p u(n+t) dt \leq C \sum_{n=0}^{\infty} \int_0^{a_n} u(n+t) dt. \quad (3)$$

Case \prec_3 : For every $n \in \mathbb{N} \cup \{0\}$ and $a \in [0, 1]$,

$$\int_a^1 \left(\frac{a}{t}\right)^p u(n+t) dt \leq C \int_0^{n+a} u(t) dt. \quad (4)$$

Remark 2.6 We observe that (2) of Corollary 2.5 is stronger than (3), which follows from the fact that the order \prec_1 is stronger than \prec_2 . Similarly for \prec_2 and \prec_3 (in fact, to see that (4) is weaker than (3), given $n \in \mathbb{N} \cup \{0\}$ and $a \in [0, 1]$, it suffices to consider the decreasing sequence $a_0 = \dots = a_{n-1} = 1$, $a_n = a$, and $a_k = 0$, $k = n+1, \dots$).

Also, if f is decreasing (for the classical order \leq), then

$$S_{\blacktriangleleft} f(x) \leq S f(x) = \frac{1}{x} \int_0^x f(t) dt.$$

Therefore, we obtain that the boundedness of S on $L_{\leq}^p(u(x) dx)$ (i.e., the classical B_p condition in [2]) is stronger than (4), since $\prec_3 = \leq$. A direct argument for this fact follows from the inequality

$$a \frac{x - [a]}{x} \geq a - [a], \quad x \geq a.$$

3 Weights in $B_p(\mathbb{R}_+^n)$

We will now show how to apply our previous result to obtain the weighted inequalities for the multidimensional Hardy operator, acting on decreasing functions. For simplicity we will only consider the case $n = 2$, the general case being an easy extension. We first introduce the following notations:

$$\begin{aligned} S_1 f(x, y) &= \frac{1}{x} \int_0^x f(s, y) ds, & S_2 f(x, y) &= \frac{1}{y} \int_0^y f(x, t) dt, \\ S f(x, y) &= \frac{1}{xy} \int_0^x \int_0^y f(s, t) dt ds = S_1(S_2 f)(x, y) = S_2(S_1 f)(x, y). \end{aligned}$$

We denote by $D_x = \{t > 0 : (x, t) \in D\}$, and $D_x^y = D \cap ([0, x] \times [0, y])$. $L_{\text{dec}}^p(u)$ is the usual cone of functions in $L^p(u)$, which are decreasing on each variable. Then, we have:

Theorem 3.1 *If $0 < p < \infty$, the following are equivalent conditions:*

- (a) $S : L_{\text{dec}}^p(u) \longrightarrow L^p(u)$.
- (b) *There exists a constant $C > 0$ such that, for every decreasing set D :*

$$\int_{\mathbb{R}_+^2 \setminus D} \frac{|D_x^y|^p}{(xy)^p} u(x, y) dx dy \leq C \int_D u(x, y) dx dy. \quad (5)$$

(c) $S_1, S_2 : L_{\text{dec}}^p(u) \longrightarrow L^p(u)$.

Proof:

That (a) implies (b) follows as usual: Taking $f = \chi_D$, and using the fact that

$$Sf(x, y) = \frac{|D_x^y|}{xy},$$

then (5) is a consequence of the hypothesis.

To show that (b) implies (c), we observe that if $(x, y) \notin D$, then

$$[0, x] \times D_x \subset D_x^y,$$

and hence, $x|D_x| \leq |D_x^y|$. Therefore,

$$\begin{aligned} \int_{\mathbb{R}_+^2 \setminus D} \frac{|D_x|^p}{y^p} u(x, y) dx dy &\leq \int_{D^c} \frac{|D_x^y|^p}{(xy)^p} u(x, y) dx dy \\ &\leq C \int_D u(x, y) dx dy, \end{aligned}$$

and the result follows from Corollary 2.3 (4). Similarly for S_1 .

(c) implies (a): Iterate and observe that $S_j f$ is decreasing if f is decreasing. \square

Remark 3.2 The iteration technique used to prove Theorem 3.1 can be extended very easily to other settings. For example, we could consider in \mathbb{N}^2 the operator

$$S(\{a_{n,m}\}_{n,m}) = \frac{1}{nm} \sum_{j=1}^n \sum_{k=1}^m a_{j,k},$$

acting on decreasing two-indexes sequences, and obtain the characterization of the boundedness of S on the weighted sequence spaces $\ell^p(\{u_{n,m}\}_{n,m})$, for general weights $\{u_{n,m}\}_{n,m}$, which improves some of the results in [3] proved only for product weights.

Condition (5) in \mathbb{R}_+^n takes the following form:

$$\int_{\mathbb{R}_+^n \setminus D} \frac{|D \cap ([0, x_1] \times \cdots \times [0, x_n])|^p}{(x_1 \cdots x_n)^p} u(x_1, \dots, x_n) dx_1 \cdots dx_n \leq C \int_D u(x_1, \dots, x_n) dx_1 \cdots dx_n,$$

which will be denoted by $u \in B_p(\mathbb{R}_+^n)$. Observe that since $|D \cap ([0, x_1] \times \cdots \times [0, x_n])| \leq x_1 \cdots x_n$, then $B_p(\mathbb{R}_+^n) \subset B_q(\mathbb{R}_+^n)$, if $p < q$.

We will now prove that, as in the one-dimensional case (see [2] for the original result and [13] for a different proof, related to the one we will use), $B_p(\mathbb{R}_+^n)$ satisfies the $p - \varepsilon$ condition.

Theorem 3.3 *If $u \in B_p(\mathbb{R}_+^n)$, $1 \leq p < \infty$, then there exists an $\varepsilon > 0$ such that $u \in B_{p-\varepsilon}(\mathbb{R}_+^n)$.*

Proof: We will only consider the case $n = 2$ ($n \geq 3$ follows similarly). Using Theorem 3.1, it suffices to show that $S_j : L_{\text{dec}}^{p-\varepsilon}(u) \rightarrow L^{p-\varepsilon}(u)$, $j = 1, 2$, and by symmetry, we may only consider the case $j = 2$. Take any decreasing set $D \subset \mathbb{R}_+^2$. Then there exists a decreasing function $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$D = \{(s, t) \in \mathbb{R}_+^2; 0 < t < h(s)\}.$$

Therefore, Corollary 2.3 (4) gives:

$$\int_0^\infty \int_{h(s)}^\infty \frac{h^p(s)}{t^p} u(s, t) dt ds \leq C \int_0^\infty \int_0^{h(s)} u(s, t) dt ds.$$

With $f = \chi_D$, we have:

$$S_2 f(s, t) = \begin{cases} 1 & \text{if } 0 < t \leq h(s), \\ \frac{h(s)}{t} & \text{if } 0 \leq h(s) < t. \end{cases}$$

Iterating we can prove that, for every $m \in \mathbb{N}$,

$$S_2^m f(s, t) = S_2 \circ \dots \circ S_2 f(s, t) = \begin{cases} 1 & \text{if } 0 < t \leq h(s), \\ \frac{h(s)}{t} \sum_{j=0}^{m-1} \frac{1}{j!} \log^j \frac{t}{h(s)} & \text{if } 0 \leq h(s) < t. \end{cases}$$

Hence, if $h(s) < t$, we have that the following inequality follows easily

$$(S_2^m f(s, t))^p \geq \left(\frac{h(s)}{t}\right)^p \frac{1}{(m-1)!} \log^{m-1} \frac{t}{h(s)},$$

and

$$\int_0^\infty \int_{h(s)}^\infty \left(\frac{h(s)}{t}\right)^p \frac{1}{(m-1)!} \log^{m-1} \frac{t}{h(s)} u(s, t) dt ds \leq C^m \int_0^\infty \int_0^{h(s)} u(s, t) dt ds.$$

Thus, taking $\sigma > \max(C, 1/p)$, and adding up in m :

$$\begin{aligned} & \int_0^\infty \int_{h(s)}^\infty \left(\frac{h(s)}{t}\right)^p \sum_{m=1}^\infty \frac{1}{\sigma^{m-1}(m-1)!} \log^{m-1} \frac{t}{h(s)} u(s, t) dt ds \\ &= \int_0^\infty \int_{h(s)}^\infty \left(\frac{h(s)}{t}\right)^{p-1/\sigma} u(s, t) dt ds \\ &\leq \sum_{m=1}^\infty \left(\frac{C}{\sigma}\right)^m \int_0^\infty \int_0^{h(s)} u(s, t) dt ds = C' \int_0^\infty \int_0^{h(s)} u(s, t) dt ds, \end{aligned}$$

and the result follows with $\varepsilon = 1/\sigma$. □

It was proved in [5] that for the case of the identity operator (i.e., when considering embeddings), one cannot, in general, replace the condition on all decreasing sets by just taking rectangles of the form $[0, a_1] \times \cdots \times [0, a_n]$, $a_j > 0$. However, in the case of product weights, both conditions were equivalent (see [5, Theorem 2.5]). We will now show that in this context, $u(x) = \prod_{j=1}^n u_j(x_j) \in B_p(\mathbb{R}_+^n)$, factorizes very nicely as $u_j \in B_p$, for all $j \in \{1, \dots, n\}$.

Theorem 3.4 *Let $u(x) = \prod_{j=1}^n u_j(x_j) \in B_p(\mathbb{R}_+^n)$ be a product weight. Then the following conditions are equivalent:*

(a) $u \in B_p(\mathbb{R}_+^n)$.

(b) For every $a_j > 0$, $j \in \{1, \dots, n\}$,

$$\begin{aligned} & \int_{\mathbb{R}_+^n \setminus ([0, a_1] \times \cdots \times [0, a_n])} \frac{|([0, a_1] \times \cdots \times [0, a_n]) \cap ([0, x_1] \times \cdots \times [0, x_n])|^p}{(x_1 \cdots x_n)^p} u(x) dx_1 \cdots dx_n \\ & \leq C \int_{[0, a_1] \times \cdots \times [0, a_n]} u(x) dx_1 \cdots dx_n. \end{aligned}$$

(c) $u_j \in B_p$, $j \in \{1, \dots, n\}$.

Proof: As before, and by simplicity, we will work the details only for $n = 2$.

If $u \in B_p(\mathbb{R}_+^2)$, then evaluating (5) for rectangles of the form $[0, a_1] \times [0, a_2]$ we get (b).

Assuming now that (b) holds, we evaluate the condition to obtain:

$$\begin{aligned} & \int_{\mathbb{R}_+^2 \setminus ([0, a_1] \times [0, a_2])} \frac{|([0, a_1] \times [0, a_2]) \cap ([0, x_1] \times [0, x_2])|^p}{(x_1 x_2)^p} u_1(x_1) u_2(x_2) dx_1 dx_2 \\ & = \left(\int_0^{a_1} u_1(x_1) dx_1 \right) a_2^p \int_{a_2}^\infty \frac{u_2(x_2)}{x_2^p} dx_2 + \left(\int_0^{a_2} u_2(x_2) dx_2 \right) a_1^p \int_{a_1}^\infty \frac{u_1(x_1)}{x_1^p} dx_1 \\ & \quad + (a_1 a_2)^p \left(\int_{a_1}^\infty \frac{u_1(x_1)}{x_1^p} dx_1 \right) \left(\int_{a_2}^\infty \frac{u_2(x_2)}{x_2^p} dx_2 \right) \\ & \leq C \left(\int_0^{a_1} u_1(x_1) dx_1 \right) \left(\int_0^{a_2} u_2(x_2) dx_2 \right), \end{aligned}$$

from which we easily deduce that, for $j = 1, 2$,

$$a_j^p \int_{a_j}^\infty \frac{u_j(x_j)}{x_j^p} dx_j \leq C \int_0^{a_j} u_j(x_j) dx_j,$$

and hence $u_j \in B_p$.

Finally, iterating the one-dimensional Hardy operator, and using the fact that u is a product weight, we deduce that

$$S : L_{\text{dec}}^p(u) \longrightarrow L^p(u),$$

which is (a). □

Remark 3.5 The equivalence between (c) of Theorem 3.4 and the boundedness of the Hardy operator $S : L_{\text{dec}}^p(u_1 u_2) \longrightarrow L^p(u_1 u_2)$, for the range $p \geq 1$, was proved in [4], by using an indirect argument related to the characterization of the normability property of some multidimensional analogs of the weighted Lorentz spaces (in particular this proof did not make use of the $B_p(\mathbb{R}_+^n)$ condition). For the case $p = 1$ one can even prove a quantitative estimate of the constant in the B_1 condition, namely, if we set

$$\|u\|_{B_1(\mathbb{R}_+^2)} = \sup_{D \text{ decreasing}} \frac{\int_{\mathbb{R}_+^2} S\chi_D(s, t)u(s, t) ds dt}{\int_D u(s, t) ds dt},$$

then, $\|u_1(x_1)u_2(x_2)\|_{B_1(\mathbb{R}_+^2)} = \|u_1\|_{B_1}\|u_2\|_{B_1}$.

As we pointed out in Remark 3.5, similar results to Theorem 3.4 can be obtained, for product weights, in more general settings (for example in \mathbb{N}^2 , see [3]).

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