AM1 – II APPELLO.

Prova Parziale (e Complessiva).

(1) $$f(x) = \frac{2e^{2x} + 2\cos(2x) - 4 - \sin(4x)}{\sin^2(4x)\cdot\log(1+3x+7x^2)}$$

$$\underset{x\to 0+}{\sim} \frac{2e^{2x} + 2\cos(2x) - 4 - \sin(4x)}{(4x)^2\cdot 3x} =$$

$$= \left\{2\left[1 + (2x) + (2x)^2/2 + (2x)^3/6 + o(x^3)\right] + 2\left[1 - \frac{(2x)^2}{2} + o(x^3)\right]\right.$$
$$\left. - 4 - \left[(4x) - (4x)^3/6 + o(x^3)\right]\right\} / (4^2\cdot 3\cdot x^3)$$

$$= \left\{2+2-4+4x-4x+4x^2-4x^2+\frac{2^3}{3}x^3+\frac{4^3}{6}x^3+o(x^3)\right\}/(2^4\cdot 3\cdot x^3)$$

$$\underset{x\to 0+}{\sim} \frac{2^3+2^5}{3\cdot 3\cdot 2^4} = \frac{5}{18} \Longrightarrow L = 5/18$$

(2) $$\int_{1/2}^{1} \frac{\log(x)}{\log^2(2x)+\log^2(2)}\cdot\frac{dx}{x} = I.$$

Pongo $y = \log(2x)$

$dy = dx/x$

$\log(x) = \log(2x) - \log 2$
$= y - \log 2$

$x = 1/2 \Rightarrow y = 0$; $x = 1 \Rightarrow y = \log 2$

Quindi $$I = \int_0^{\log(2)} \frac{y - \log 2}{y^2+\log^2 2}\, dy$$

$$= \frac{1}{2}\int_0^{\log(2)} \frac{2y\, dy}{y^2+\log^2 2} - \log 2\cdot\int_0^{\log 2}\frac{dy}{\left[\left(\frac{y}{\log 2}\right)^2+1\right]\cdot\log^2 2}$$

$$= \frac{1}{2}\cdot\left[\log\left(y^2+\log^2 2\right)\right]_{y=0}^{y=\log 2} - \frac{\cancel{\log 2}}{\cancel{\log 2}}\cdot\left[\operatorname{arctg}\left(\frac{y}{\log 2}\right)\right]_{y=0}^{y=\log 2}$$

$$= \frac{1}{2}\log\left(\frac{2\log^2 2}{\log^2 2}\right) - \operatorname{arctg}(1) = \frac{\log 2}{2} - \frac{\pi}{4}$$

(3) $f(x) = \left[|x^2-4| + (x^2-4)\right] \cdot e^{-|x-2|} + 3$; $f \in C(\mathbb{R})$.

- Chiaramente, $\lim_{x \to \pm\infty} f(x) = 0 + 3 = 3$

- f è derivabile almeno se $x-2 \neq 0$ $(x \neq 2)$ e $x^2-4 \neq 0$ $(x \neq \pm 2)$:

  $\exists\, f'(x)$ se $x \in \mathbb{R} \setminus \{-2, 2\}$.

Calcolo: $f'(x) = e^{-|x-2|} \cdot \left\{ 2x \cdot \operatorname{sgn}(x^2-4) + 2x - (|x^2-4| + x^2-4)\operatorname{sgn}(x-2) \right\}$

$= 2 \cdot e^{-|x-2|} \cdot \left\{ x\left(\operatorname{sgn}(x^2-4)+1\right) - (|x^2-4| + x^2-4) \cdot \operatorname{sgn}(x-2) \right\}$

- Osservo che, se $x^2-4 < 0$ (cioè $-2 < x < 2$), allora

  $f'(x) = 2 \cdot e^{-|x-2|} \cdot \left\{ x(-1+1) - (-x^2+4+x^2-4) \cdot \operatorname{sgn}(x-2) \right\} = 0$.

cioè, f è costante in $]-2, 2[$, quindi in $[-2, 2]$ per continuità: $\boxed{x \in [-2, 2] \Rightarrow f(x) = f(2) = 3}$

- Se $x^2-4 > 0$ (cioè, se $-2 > x$ o $x > 2$), allora

  $f'(x) = 2 \cdot e^{-|x-2|} \cdot \left\{ 2x - 2(x^2-4) \cdot \operatorname{sgn}(x-2) \right\}$

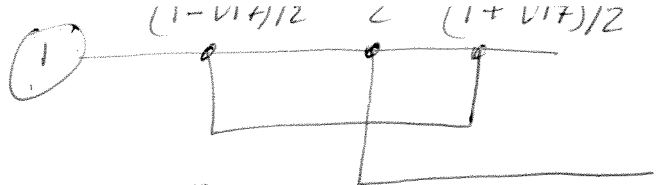
  $= 4 \cdot e^{-|x-2|} \cdot \left\{ x - (x^2-4) \cdot \operatorname{sgn}(x-2) \right\}$

Per vedere se f cresce/decresce, ~~studio~~ cerco x t.c. $x^2-4 > 0$ e $f'(x) > 0$. Ho i casi:

$$(1) \begin{cases} x > 2 \\ f'(x) = 4 \cdot e^{-(x-2)} \cdot \{x - (x^2-4) > 0\} \end{cases} \quad \begin{cases} x > 2 \\ x^2 - x - 4 < 0 \end{cases}$$

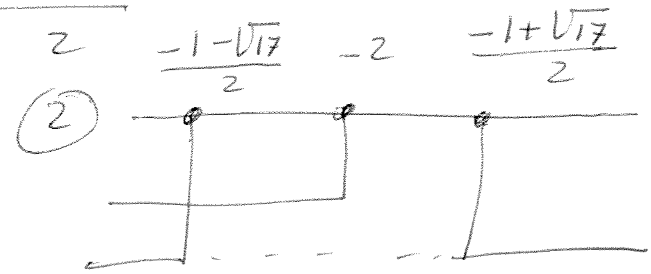
$$(2) \begin{cases} x < -2 \\ f'(x) = 4 \cdot e^{-(2-x)} \cdot \{x + x^2 - 4 > 0\} \end{cases} \quad \begin{cases} x < -2 \\ x^2 + x - 4 > 0 \end{cases}$$

$$(1)\quad \begin{cases} x > 2 \\ \dfrac{1-\sqrt{17}}{2} < x < \dfrac{1+\sqrt{17}}{2} \end{cases}$$



$$2 < x < \frac{1+\sqrt{17}}{2}$$

$$(2)\quad \begin{cases} x < -2 \\ x < \dfrac{-1-\sqrt{17}}{2} \ \text{ o } \ x > \dfrac{-1+\sqrt{17}}{2} \end{cases}$$



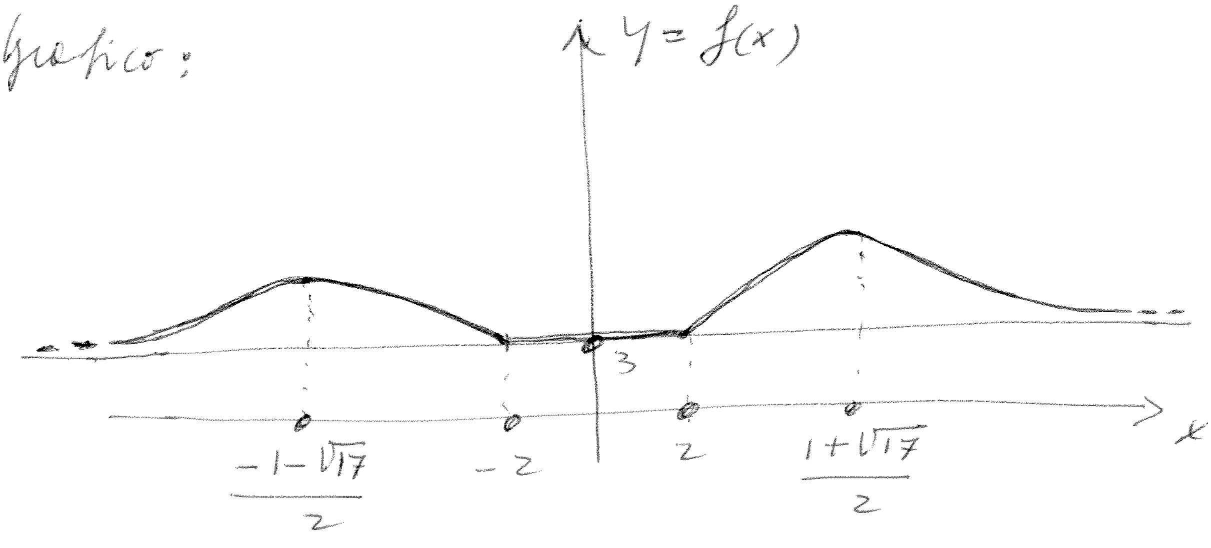
$$x < \frac{-1-\sqrt{17}}{2}$$

Per continuità, f è crescente su $\left]-\infty, \frac{-1-\sqrt{17}}{2}\right]$ e su $\left[2, \frac{1+\sqrt{17}}{2}\right]$

f è decrescente su $\left[\frac{-1-\sqrt{17}}{2}, -2\right]$ e su $\left[\frac{1+\sqrt{17}}{2}, +\infty\right[$

e costante su $[-2, 2]$.

Grafico:



$f'(2^+) = 4 \cdot e^0 \cdot \{2\} > 0 = f'(2^-)$

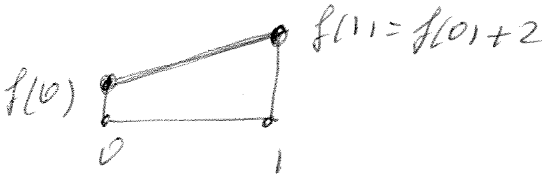
$f'(-2^-) = 4 \cdot e^{-4} \cdot (-2) < 0 = f'(-2^+)$ $\Rightarrow$ f non è derivabile in 2 e in -2.

(4) Per il Teorema di Lagrange, di cui sussistono le ipotesi,

$$\exists x \in]0,1[\text{ t.c. } f'(x) = \frac{f(1)-f(0)}{1-0} = \frac{f(0)+2-f(0)}{1-0} = 2 > 0:$$

vale la seconda risposta.

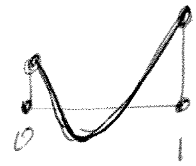
Non la prima: $g'(x) = f'(x)\ \forall x \in]0,1[$ e potrebbe essere



$f'(x) \neq 0\ \forall x \in]0,1[$.

Non la terza: lo stesso grafico lo mostra bene.

Non la quarta:



(5) $h(x) = \log\left(f(e^{2x})\right)^3 = 3\log\left(f(e^{2x})\right)$

$$\Rightarrow h'(x) = 3 \cdot \frac{f'(e^{2x}) \cdot e^{2x} \cdot 2}{f(e^{2x})} = \frac{6 \cdot f'(e^{2x}) \cdot e^{2x}}{f(e^{2x})}$$

$$\Rightarrow h'(1) = \frac{6 \cdot f'(e^2) \cdot e^2}{f(e^2)}$$

(3) $z^5 = -2i = 2 \cdot e^{-i\pi/2} \iff \boxed{z = \sqrt[5]{2} \cdot \left[\cos\left(-\frac{\pi}{10} + \frac{2k\pi}{5}\right) + i\sin\left(-\frac{\pi}{10} + \frac{2k\pi}{5}\right)\right] \text{ con } k = 0,1,2,3,4}$

$z^2 - 6iz - 5 = 0$

$$\frac{\Delta}{4} = (-3i)^2 + 5 = -9 + 5 = -4 = (2i)^2$$

$$z = 3i \pm 2i \qquad \boxed{z = 5i,\ z = i}$$