ANALISI MATEMATICA I; 9/9/2010

Es. 1

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + o(x^3)_{x\to 0}$$

$$\frac{1}{1-\sin(x)} = 1 + \sin(x) + \sin^2(x) + \sin^3(x) + o(\sin^3(x))_{x\to 0}$$

$$= 1 + x - \frac{x^3}{6} + o(x^3) + (x + o(x^2))^2 + (x + o(x^2))^2 + o(x^3)$$

$$= 1 + x + x^2 + x^3 - x^3/6 + o(x^3)_{x\to 0}$$

Ho usato il fatto che $\sin(x) \sim_{x\to 0} x$.

$$\Rightarrow \frac{1}{1-x} - \frac{1}{1-\sin(x)} = 1 + x + x^2 + x^3 - \left(1 + x + x^2 + x^3 - \frac{x^3}{6}\right) + o(x^3)$$

$$= \frac{x^3}{6} + o(x^3)$$

Ho che $\frac{1}{1-x} + \frac{1}{1-\sin(x)} \xrightarrow[x\to 0]{} 2$,

quindi, se $f(x)$ è la funzione di cui calcolo il limite,

$$f(x) = \frac{\frac{x^3}{6} + o(x^3)}{\frac{x^3}{2}(2 + o(1))} = \frac{\frac{1}{6} + o(1)}{\frac{1}{2}(2 + o(1))} \xrightarrow[x\to 0]{} \frac{1}{6}$$

$L = 1/6$

Es. 2

$$I = \int_0^{1/2} (x+1)\, e^x \cdot (e^{2x} + 1)\, dx$$

$$= \int_0^{1/2} (x+1)\cdot(e^{3x} + e^x)\, dx = \text{(integro per parti)}$$

$$= \left[\left(\frac{e^{3x}}{3} + e^x\right)\cdot(x+1)\right]_0^{1/2} - \int_0^{1/2} \left(\frac{e^{3x}}{3} + e^x\right)\cdot 1 \cdot dx$$

$$= \left[\left(\frac{e^{3x}}{3} + e^x\right)(x+1)\right]_0^{1/2} - \left(\frac{e^{3x}}{9} + e^x\right)_0^{1/2}$$

$\boxed{\text{Es. 3}}$ $\quad f(x) = \left| e^{|x^2-1|} - e \right|$

$\lim\limits_{x \to \pm\infty} f(x) = +\infty$

$f'(x) = \mathrm{sgn}\left(e^{|x^2-1|} - e\right) \cdot e^{|x^2-1|} \cdot \mathrm{sgn}(x^2-1) \cdot 2x$

esiste (al meno) se $x \neq \pm 1$ e $|x^2-1| \neq 1$,

cioè se $x \neq \pm 1$, $x^2 - 1 \neq \pm 1$

$\mathrm{Dom}(f') \supseteq \mathbb{R} \setminus \{1, -1, 0, \pm\sqrt{2}\}$

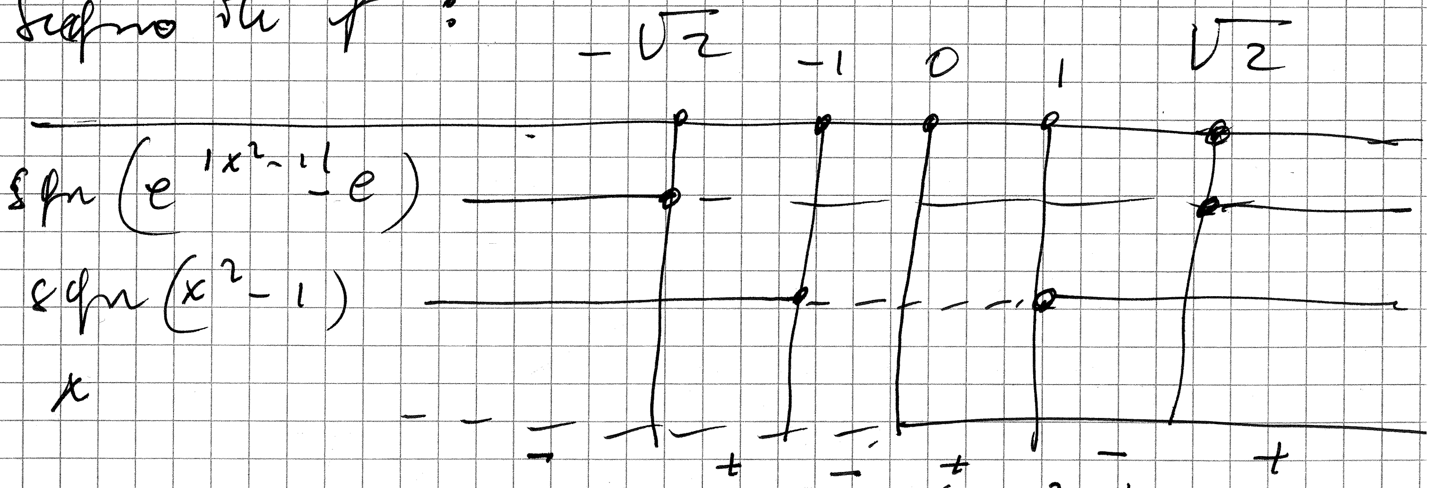
$f \in C(\mathbb{R})$.

~~Ma~~: Poiché $\exists \lim\limits_{x \to 0} f'(x) = 0$, $\exists f'(0) = 0$.

$\mathrm{Dom}(f') = \mathbb{R} \setminus \{1, -1, \pm\sqrt{2}\}$

$f(0) = 0$

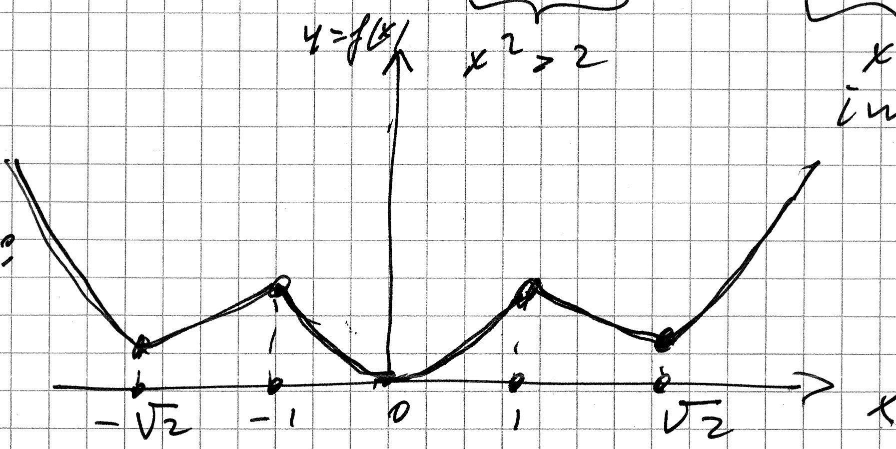
Segno di f':



Ho usato il fatto che $\mathrm{sgn}\left(e^{|x^2-1|} - e\right) > 0$

sse $|x^2-1| \geq 1$ sse $\underbrace{x^2 - 1 > 1}_{x^2 > 2}$ o $\underbrace{x^2 - 1 < -1}_{x^2 < 0,\ \text{impossibile}}$

Grafico:



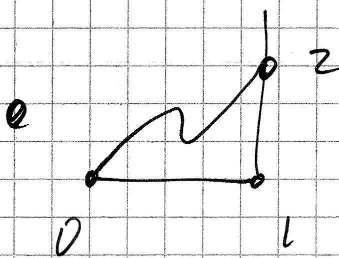
Es. 4 ~~Poiché~~ Dalle ipotesi su f segue che (per Lagrange):

$\exists x \in]0,1[$:

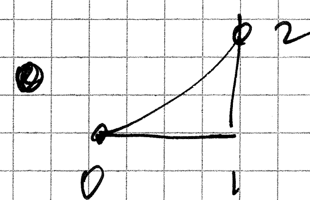
$$f'(x) = \frac{f(1) - f(0)}{1 - 0} = \frac{2}{1} = 2 > 0,$$

quindi $\boxed{\exists x \in]0,1[: f'(x) > 0.}$

Per ciascuna delle altre affermazioni si trovano dei controesempi:

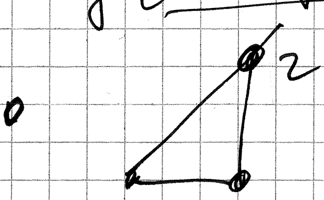
- 

  f non è necessariamente crescente

- 

  f potrebbe essere crescente, quindi non esisterebbe $x \in]0,1[: f'(x) < 0$

- ~~Se f fosse strettamente crescente, lo sarebbe anche $g(x) = f(x) + 2$, e~~

- 

  Prendo $f(x) = 2x$,
  $g(x) = 2x + 1$
  e $g'(x) = 2 \neq 0 \quad \forall x$.

Es. 5 Per il Teorema fondamentale del Calcolo Integrale,

$$\exists \; h'(x) = f(0).$$

Es. 6 $z^5 = -3 = 3 \cdot e^{\pi i} \iff z = \sqrt[5]{3} \cdot \left[\cos\left(\frac{\pi + 2k\pi}{5}\right) + i \sin\left(\frac{\pi + 2k\pi}{5}\right)\right]$,
$k = 0, 1, 2, 3, 4$.

$\Delta = (10i)^2 - 34 = -134$: $z = \dfrac{-10i \pm \sqrt{134}\, i}{2}$