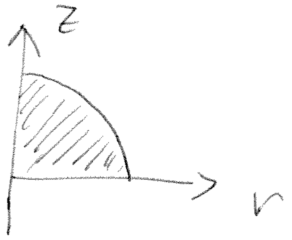
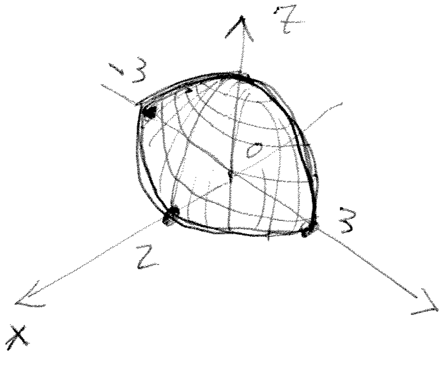
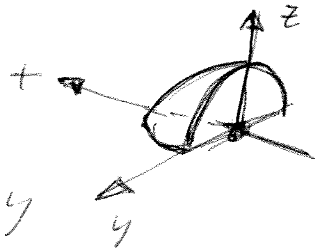
AM2 — II APPELLO

Prove Parziali

(1) Per comodità pongo $\begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \end{cases}$ con $\begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 1 \\ r^2 + z^2 \le 1 \\ z \ge 0 \end{cases}$







Uso il Teorema della divergenza con

$$\operatorname{div} F(x,y,z) = 3z e^{-z^2} + 2z e^{-z^2} + 0 = 5z e^{-z^2},$$ quindi

$$\iint_{\partial\Omega} F \cdot \nu \, d\sigma = \iiint_{\Omega} 5z \cdot e^{-z^2} \, dx\, dy\, dz$$

$$= 6 \cdot \int_{-\pi/2}^{\pi/2} d\theta \int_0^1 r\, dr \int_0^{\sqrt{1-r^2}} dz \; 5z e^{-z^2} = 5\pi \cdot 6 \cdot \int_0^1 r\, dr \cdot \left(\frac{e^{-z^2}}{-2} \right)_{z=0}^{z=\sqrt{1-r^2}}$$

$$= 6 \cdot \frac{5}{2}\pi \cdot \int_0^1 \left(-e^{-(1-r^2)} + 1 \right) r\, dr = 6 \cdot \frac{5}{2}\pi \cdot \left(\frac{r^2}{2} - e^{-1} \frac{e^{r^2}}{2} \right)_0^1$$

$$= 6 \cdot \frac{5}{4}\pi \cdot \left[(1-1) - (0 - e^{-1}) \right] = 6 \cdot \frac{5}{4}\pi \cdot e^{-1} = \frac{15}{2} \cdot \pi \cdot e^{-1}$$

(2) Σ è la parte "curva" di $\partial\Omega$, orientata secondo la normale interna a Ω:

Uso il teorema del rotore e una parametrizzazione di Σ.

$$\mathrm{Rot}\,G(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & 0 \end{vmatrix}$$

$$= (x, y, -2z).$$

Parametrizzo Σ mediante $(x,y) \xmapsto{\Phi} \left(x, y, \sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\right)$

$\Phi : A \to \mathbb{R}^3$, $A = \{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \le 1,\ x \ge 0\} \subseteq \mathbb{R}^2$

$$(\partial_x \Phi \times \partial_y \Phi)(x,y) = \begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{x}{4}\Big/\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}} \\ 0 & 1 & -\frac{y}{9}\Big/\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}} \end{vmatrix}$$

$$= \left(\frac{x}{4\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},\ \frac{y}{9\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}},\ 1 \right)$$

e $(\partial_x \Phi \times \partial_y \Phi)(0,0) = (0,0,1) \neq \mu(0,0,1)$:

Φ non è compatibile con l'orientazione μ. Quindi

$$\int_{(\partial\Sigma,\mu)} G(w)\cdot dw = \iint_\Sigma \mathrm{Rot}\,G\cdot\mu \, d\sigma =$$

$$= -\iint_A \frac{\frac{x^2}{4} + \frac{y^2}{9} - 2\left(\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}\right)^2}{\sqrt{1-\frac{x^2}{4}-\frac{y^2}{9}}}\, dx\, dy$$

Nelle coordinate ellittiche precedenti:

$$= -6\int_{-\pi/2}^{\pi/2} d\theta \cdot \int_0^1 r\,dr \cdot \left(\frac{r^2}{\sqrt{1-r^2}} - 2\cdot\sqrt{1-r^2}\right)$$

$$= -6\cdot\pi\cdot\left\{\frac{1}{2}\int_0^1 \frac{t}{\sqrt{1-t}}\,dt - \int_0^1 \sqrt{1-t}\,dt\right\}$$

$$= -6\cdot\pi\cdot\left\{\frac{1}{2}\int_0^1 \frac{t-1+1}{\sqrt{1-t}}\,dt + \left[\frac{(1-t)^{3/2}}{3/2}\right]_0^1\right\}$$

$$= -6\pi\left\{\frac{1}{2}\int_0^1\left(-\sqrt{1-t} + \frac{1}{\sqrt{1-t}}\right)dt - \frac{2}{3}\right\}$$

$$= -6\pi\left\{\frac{1}{2}\cdot\left[\frac{(1-t)^{3/2}}{3/2}\right]_0^1 - \frac{1}{2}\left[\frac{(1-t)^{1/2}}{1/2}\right]_0^1 - \frac{2}{3}\right\}$$

$$= -6\pi\cdot\left\{\frac{1}{2}\left(-\frac{2}{3}\right) - \frac{1}{2}\cdot(-2) - \frac{2}{3}\right\}$$

$$= -6\pi\cdot\left(-\frac{2}{3}\right) = 12\pi$$

(3) Risolvo $z'' + 4z = 0$.

$\lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$

$z(x) = A\cos(2x) + B\sin(2x)$: ho risonanza.

Pongo $y(x) = x\left[A\cos(2x) + B\sin(2x)\right]$

$y'(x) = A\cos(2x) + B\sin(2x) + x\cdot\left[-2A\sin(2x) + 2B\cos(2x)\right]$

$y''(x) = 2\left[-2A\sin(2x) + 2B\cos(2x)\right] + x\left[-4A\cos(2x) - 4B\sin(2x)\right]$

Sostituendo in $y'' + 4y = 3\sin(2x)$ e facendo i conti ho:

$$4\left(-A\cdot\sin(2x)+B\cdot\cos(2x)\right)=3\sin(2x):$$

$$B=0\,,\quad A=-\frac{3}{4}.$$

L'integrale generale è quindi:

$$y(x)=\alpha\cdot\cos(2x)+\beta\sin(2x)-\frac{3}{4}x\cos(2x)$$

$$y'(x)=-2\alpha\cdot\sin(2x)+2\beta\cos(2x)+\frac{3}{2}x\sin(2x)-\frac{3}{4}\cos(2x$$

$$y(0)=0 \iff \alpha-\frac{3}{4}=0 \qquad \alpha=\frac{3}{4}$$

$$y'(0)=1 \iff 2\beta=3/4 \qquad \beta=3/8$$

$$\boxed{y(x)=\cancel{\frac{3}{4}\cos(2x)}+\frac{3}{8}\sin(2x)-\frac{3}{4}\cos(2x)}$$

è la soluzione del problema di Cauchy.

<u>Prova scritta complessiva</u>:

$$f(x,y)=x\cdot e^{-9x^2-4y^2}+3$$

$$\begin{cases} f_x=e^{-9x^2-4y^2}\cdot\{1-18x^2\} \\ f_y=-8xy \end{cases}$$

$$\nabla f(x,y)=0 \iff \begin{cases} xy=0 \\ 1-18x^2=0 \end{cases}$$

$$\begin{cases} y=0 \\ x=\pm\frac{1}{\sqrt{18}}=\pm\frac{1}{3\sqrt{2}} \end{cases}$$

$$f_{yy}=-8x\cdot e^{-9x^2-4y^2}$$

$$f_{yx}=-8y\cdot e^{-9x^2-4y^2}$$

$$f_{xx}=e^{-9x^2-4y^2}\cdot\{-36x-18x\cdot(1-18x^2)\}$$

$$=-18x\cdot e^{-9x^2-4y^2}\cdot(3-18x^2)$$

$$\text{Hess} f\left(\pm\frac{1}{\sqrt{18}},0\right) = e^{-9/8}\cdot\begin{bmatrix}\mp 2\cdot\sqrt{18} & 0\\ 0 & \mp\frac{8}{\sqrt{18}}\end{bmatrix}$$

è def. pos. in $\left(-\frac{1}{\sqrt{18}},0\right) \to$ p.to di ~~max~~ min. relativo

è def. neg. in $\left(\frac{1}{\sqrt{18}},0\right) \to$ p.to di MAX. relativo

- $\nabla f(0,3) = \left(e^{-36}, 0\right)$; $f(0,3) = 3$

Piano tangente : $Z = e^{-36}\cdot x + 3$

Spazio tangente : $Z = e^{-36}\cdot x$

(4) Nelle coordinate ellittiche di cui sopra:

$$\iint_A \left(1 - e^{-9x^2-4y^2}\right)dx\,dy = 6\cdot\int_{-\pi/2}^{\pi/2} d\theta \int_0^1 r\,dr\left(1-e^{-36r^2}\right)$$

$$= 6\cdot\pi\cdot\left[\frac{1}{2} - \left(\frac{e^{-36r^2}}{-36\cdot 2}\right)\Big|_0^1\right]$$

$$= 6\cdot\pi\cdot\left[\frac{1}{2} + \frac{e^{-36}-1}{2\cdot 36}\right] = \frac{6\cdot\pi}{2\cdot 36}\left(36-1+e^{-36}\right)$$

$$= \frac{\pi}{12}\left(35+e^{-36}\right).$$

(5) ~~Dunque~~ Per definizione di rotore:

$$h(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -y\, f(9x^2+4y^2) & x\, f(9x^2+4y^2) & \sin(6z) \end{vmatrix}$$

$$= i\,(0) + j\,(0) + k \cdot \Big[f(9x^2+4y^2) + 18x^2 f'(9x^2+4y^2)$$
$$+ f(9x^2+4y^2) + 8y^2 \cdot f'(9x^2+4y^2) \Big]$$

$$= \Big(0, 0, 2 \cdot \big[f(9x^2+4y^2) + (9x^2+4y^2) \cdot f'(9x^2+4y^2) \big] \Big)$$

$$\Rightarrow h(2,3,\pi/4) = \big(0, 0, 2\,[f(72) + 72 \cdot f'(72)]\big).$$