Prova scritta di Analisi Matematica II (9/1/2013)

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Prova orale: inizio appello/fine appello (cancellare se non interessa), non nella mezza giornata di..

(1) **[14 pti]** Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \left\{(x, y, z) : \frac{1}{4} \leq \frac{x^2}{9} + \frac{y^2}{36} \leq z^2 + 1 \leq 5 \right\}$.

(1.1) Fare un disegno *qualitativo* di Ω.

(1.2) Parametrizzare $\partial\Omega$ e dire se le parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$. (Nella formula devono apparire, magari iterati, solo integrali di una variabile).

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x, y, z) = (y, -x, z)$.

(1.5) . Sia $\Sigma = \left\{(x, y, z) : \frac{x^2}{9} + \frac{y^2}{36} = z^2 + 1,\ 0 \leq z \leq 2\right\}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la normale ν a Σ (ν essendo la normale di cui al punto (1.2)).

(1.6) Calcolare $\iint_\Sigma (\nabla \times F) \cdot \cancel{\nu} N \, d\sigma$, con la stessa F di (1.4).

(2) [2 pti] Dire per quale valore del parametro $\alpha \in \mathbb{R}$ il campo $F_\alpha : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ è chiuso, dove

$$F_\alpha(x,y) = \left(\frac{-\alpha y}{x^2+y^2} + y, \frac{x}{x^2+y^2} + x \right).$$

Sia γ la circonferenza di centro $(0,0)$ e raggio 1 e sia F l'unico campo chiuso tra i campi F_α. Calcolare

$$\int_\gamma F(z) \cdot dz.$$

Dire se F è esatto.

(3) [5 pti] Sia $A = \left\{ (x,y) : \frac{(x+y)^2}{9} + \frac{(x-y)^2}{36} \le 1 \right\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A e^{\frac{(x+y)^2}{9} + \frac{(x-y)^2}{36}} \, dxdy.$$

(4) [3 pti] Trovare l'integrale generale di

$$y'' - 36y = \cos(6x) + e^{-6x}$$

(5) [2 pti] Sia $\alpha \in C^1(\mathbb{R}, \mathbb{R})$ e si definisca il campo vettoriale $F : \mathbb{R}^3 \to \mathbb{R}^3$,

$$F(x, y, z) = (y \cdot \alpha(x^2 + y^2), x \cdot \alpha(x^2 + y^2), \alpha(z)).$$

Calcolare il rotore $\nabla \times F$ di F in (x, y, z).

(6) [4 pti] Classificare i punti critici di $f(x, y) = \left(\frac{(x+y)^2}{9} + \frac{(x-y)^2}{36} - 1\right) \cdot x + 5$.

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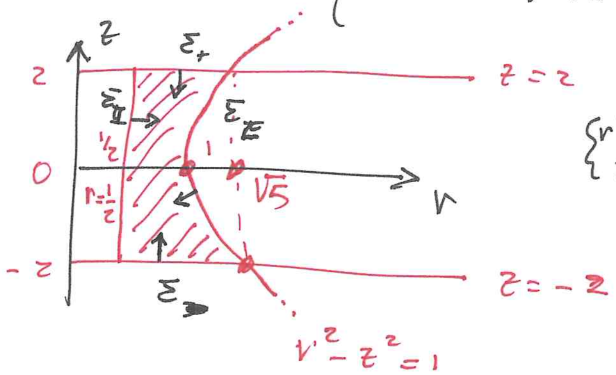
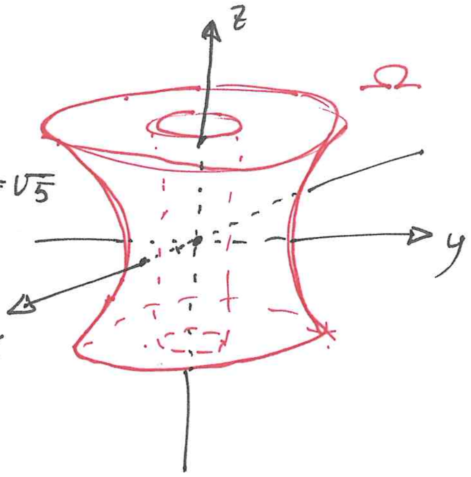
(1.1) $\begin{cases} \dfrac{x}{3} = r\cos\theta \\ \dfrac{y}{6} = r\sin\theta \end{cases}$ con $r \geq 0$, $|\theta| \leq \pi$: $(x,y,z) \in \Omega \iff \begin{cases} r \geq 0;\ |\theta| \leq \pi; \\ \frac{1}{4} \leq r^2 \leq z^2+1 \leq 5 \end{cases}$

$\iff \begin{cases} r \geq 0;\ |\theta| \leq \pi;\ r \geq \frac{1}{2};\ r^2 - z^2 \leq 1; \\ |z| \leq 2 \end{cases}$



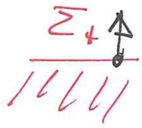
$\begin{cases} r^2 - z^2 = 1 \\ z = 2 \end{cases} \Rightarrow r = \sqrt{5}$



(1.2) • $\Sigma_+ = \{(x,y,2):\ \frac{1}{4} \leq \frac{x^2}{9} + \frac{y^2}{36} \leq 5\} = \Phi_+(A_+)$ con

$\mathbb{R}^2 \supseteq A_+ = \{(x,y):\ \frac{1}{4} \leq \frac{x^2}{9} + \frac{y^2}{36} \leq 5\} \xrightarrow{\Phi_+} \mathbb{R}^3$; $\Phi_+(x,y) = (x,y,2)$

$\partial_x \Phi_+ \times \partial_y \Phi_+ = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$: comp. con ν.



• $\Sigma_- = \{(x,y,-2):\ \frac{1}{4} \leq \frac{x^2}{9} + \frac{y^2}{36} \leq 5\} = \Phi_-(A_-)$ con

$\mathbb{R}^2 \supseteq A_- = A_+$ e $\Phi_-(x,y) = (x,y,-2)$; $\partial_x \Phi_- \times \partial_y \Phi_- = (0,0,1)$:

non comp. con ν

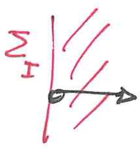
• $\Sigma_I = \{(x,y,z):\ \frac{x^2}{9} + \frac{y^2}{36} = \frac{1}{4};\ |z| \leq 2\} = \Phi_I(A_I)$ con

$A_I = \{(\theta,z): |\theta| \leq \pi, |z| \leq 2\} = [-\pi,\pi] \times [-2,2] \xrightarrow{\Phi_I} \mathbb{R}^3$;

$\Phi_I(\theta,z) = (\frac{3}{2}\cos\theta, \frac{6}{2}\sin\theta, z)$

$\partial_\theta \Phi_I \times \partial_z \Phi_I = \begin{vmatrix} i & j & k \\ -\frac{3}{2}\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\theta, \frac{3}{2}\sin\theta, 0)$

Per $\underset{\substack{\downarrow \\ \text{semipiano} \\ y=0,\ x\geq 0}}{\theta = 0}$: $\partial_\theta \Phi_I \times \partial_z \Phi_I = (3,0,0)$ non comp. con ν.



• $\Sigma_E = \{(x,y,z):\ \frac{x^2}{9} + \frac{y^2}{36} = z^2+1;\ |z| \leq 2\} = \Phi_E(A_E)$ con

$\mathbb{R}^2 \supseteq A_E = \{(z,\theta): |z| \leq 2, |\theta| \leq \pi\} = [-2,2] \times [-\pi,\pi] \xrightarrow{\Phi_E} \mathbb{R}^3$; $\Phi_E(z,\theta) = (3\sqrt{z^2+1}\cdot\cos\theta, 6\sqrt{z^2+1}\cdot\sin\theta, z)$

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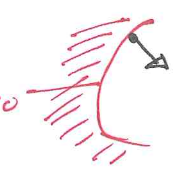
$$\partial_z \Phi_E \times \partial_\theta \Phi_E = \begin{vmatrix} i & j & k \\ -3\sqrt{z^2+1}\cdot\sin\theta & 6\sqrt{z^2+1}\cdot\cos\theta & 0 \\ \frac{3z\cos\theta}{\sqrt{z^2+1}} & \frac{6z\sin\theta}{\sqrt{z^2+1}} & 1 \end{vmatrix} = \left(6\sqrt{z^2+1}\cdot\cos\theta;\ 3\sqrt{z^2+1}\cdot\sin\theta,\ -36z\right)$$

Se $z>0$, $\partial_z \Phi_E \times \partial_\theta \Phi_E$ punta verso il basso:

compatibile con ν.



(1.3) Utilizzando (1.2) abbiamo $\iint_{\partial\Omega} F\circ\nu\, d\sigma =$

Σ_+ $= \int_{1/2}^{\sqrt5} dr \cdot \int_{-\pi}^{\pi} d\theta \cdot F(3r\cos\theta, 6r\sin\theta, 2)\circ(0,0,1)\cdot 18r$

Σ_- $- \int_{1/2}^{\sqrt5} dr \cdot \int_{-\pi}^{\pi} d\theta \cdot F(3r\cos\theta, 6r\sin\theta, -2)\circ(0,0,1)\cdot 18r$

Su $A_+ = A_-$ uso coordinate $\begin{cases} x = 3r\cos\theta \\ y = 6r\sin\theta \end{cases}$ $\frac12 \le r \le \sqrt5$, $|\theta|\le\pi$

$dx\cdot dy = 18r\, dr\cdot d\theta$

Σ_I $- \int_{-\pi}^{\pi} d\theta \int_{-2}^{2} dz \cdot F\left(\frac32\cos\theta, \frac62\sin\theta, z\right)\circ\left(3\cos\theta, \frac32\sin\theta, 0\right)$

Σ_E $+ \int_{-2}^{2} dz \int_{-\pi}^{\pi} d\theta \cdot F(3\sqrt{z^2+1}\cdot\cos\theta, 3\sqrt{z^2+1}\cdot\sin\theta, z)\circ(6\sqrt{z^2+1}\cdot\cos\theta, 3\sqrt{z^2+1}\sin\theta, -36z)$

Posso anche usare il T. della div., le coordinate cilindriche in (1.1) e $dx\, dy\, dz = \left|\det\left[J\begin{pmatrix} x, y, z \\ r, \theta, z\end{pmatrix}\right]\right| \cdot dr\, d\theta\, dz = 18r\, dr\cdot d\theta\cdot dz$:

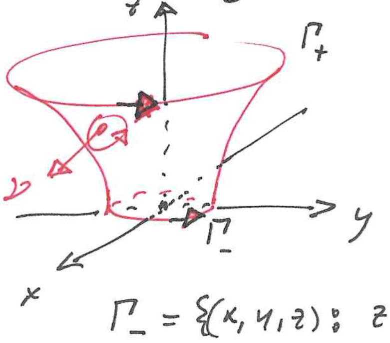
$$\boxed{\iint_{\partial\Omega} F\circ\nu\, d\sigma = \iiint_\Omega \operatorname{div} F\, dx\, dy\, dz = \int_{-\pi}^{\pi} d\theta \int_{-2}^{2} dz \int_{1/2}^{\sqrt{z^2+1}} dr\cdot \operatorname{div}F(3r\cos\theta, 6r\sin\theta, z)\cdot 18r}$$

(1.4) $\operatorname{div} F(x,y,z) = 1 \Rightarrow \iint_{\partial\Omega} F\circ\nu\, d\sigma \overset{(1.3)}{=} \int_{-\pi}^{\pi} d\theta \cdot \int_{-2}^{2} dz \cdot \int_{1/2}^{\sqrt{z^2+1}} 18r\, dr$

$= 2\pi\cdot \int_{-2}^{2} \left[9r^2\right]_{1/2}^{\sqrt{z^2+1}} dz = 18\pi \cdot \int_{-2}^{2} \left(z^2 + \frac34\right) dz = 18\pi\cdot\left[\frac{z^3}{3} + \frac34 z\right]_{-2}^{2}$

$= 18\cdot\pi\cdot\left[\frac23\cdot 2^3 + 1\right] = 18\cdot\pi\cdot\frac{19}{3} = \boxed{114\cdot\pi}$

(1.5) $\Sigma = \Sigma_E \cap \{(x,y,z):\ z\ge 0\}$



$\partial\Sigma = \Gamma_+ \cup \Gamma_-$

$\Gamma_+ = \{(x,y,z):\ z = 2,\ \frac{x^2}{9} + \frac{y^2}{36} = 5\} = \gamma_+([-\pi,\pi])$ se

$[-\pi,\pi] \xrightarrow{\gamma_+} \mathbb{R}^3$ $\gamma_+(\theta) = (3\cdot\sqrt5\cos\theta, 6\cdot\sqrt5\cdot\sin\theta, 2)$

$\dot\gamma_+(0) = (0, 6\sqrt5, 0)$: non comp. con ν.

$\Gamma_- = \{(x,y,z):\ z = 0,\ \frac{x^2}{9} + \frac{y^2}{36} = 1\} = \gamma_-([-\pi,\pi])$ se

$[-\pi,\pi] \xrightarrow{\gamma_-} \mathbb{R}^3$, $\gamma_-(\theta) = (3\cos\theta, 6\sin\theta, 0)$

$\dot\gamma_-(0) = (0, 6, 0)$: compo con ν

(106) Posso fare un calcolo diretto senza usare T. Stokes se utilizzo la parametrizzazione di Σ_E (con $z \geq 0$):

$$\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y & -x & z \end{vmatrix} = (0, 0, -2) \text{ , quindi per (103):}$$

$$\iint_\Sigma (\nabla \times F) \circ \nu \, d\sigma = + \int_0^2 dz \cdot \int_{-\pi}^{\pi} d\theta \cdot (0,0,-2) \circ (6\sqrt{z^2+1} \cdot \cos\theta, 3\sqrt{z^2+1} \cdot \sin\theta, -36z)$$

$$= \int_0^2 dz \int_{-\pi}^{\pi} d\theta \cdot 72 \cdot z = 144\pi \cdot \left[\frac{z^2}{2}\right]_0^2 = \boxed{288 \cdot \pi}$$

(2) Se $F_\alpha = (P, Q)$, allora $\partial_y P(x,y) = 1 - \alpha \cdot \frac{(x^2+y^2) - y \cdot 2y}{(x^2+y^2)^2} = 1 - \alpha \cdot \frac{x^2-y^2}{(x^2+y^2)^2}$

$\alpha = 1 \iff \|$

$$\partial_x Q(x,y) = 1 + \frac{x^2+y^2-x2x}{x^2+y^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$F(x,y) = \left(\frac{-y}{x^2+y^2} + y \, , \, \frac{x}{x^2+y^2} + x\right)$ è chiuso.

Parametrizzo γ mediante $[-\pi, \pi] \xrightarrow{\varphi} \mathbb{R}^2$; $t \mapsto \varphi(t) = (\cos t, \sin t)$

quindi $\dot{\varphi}(t) = (-\sin t, \cos t)$ e

$$\int_\gamma F(z) \circ dz = \int_{-\pi}^{\pi} F(\cos t, \sin t) \circ (-\sin t, \cos t) \, dt =$$

$$= \int_{-\pi}^{\pi} (-\sin t + \sin t, \cos t + \cos t) \circ (-\sin t, \cos t) \, dt = 2 \cdot \int_{-\pi}^{\pi} \cos^2 t \, dt = 2\pi \neq 0.$$

F non è esatto.

(3) Posto $\begin{cases} u = \frac{x+y}{3} \\ v = \frac{x-y}{6} \end{cases}$ $\qquad J\begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/6 & -1/6 \end{bmatrix} \xmapsto{\det} -\frac{2}{18} = -\frac{1}{9}$

$du \, dv = \frac{1}{9} \cdot dx \, dy \Rightarrow dx \, dy = 9 \cdot du \, dv$

$(x,y) \in A \iff (u,v) \in B = \{(u,v) : u^2 + v^2 \leq 1\}$:

$$\iint_A e^{(x+y)^2/9 + (x-y)^2/36} \, dx \, dy = 9 \cdot \iint_B e^{u^2+v^2} \, du \, dv$$

$$= 9 \cdot \int_{-\pi}^{\pi} d\theta \cdot \int_0^1 e^{r^2} \cdot r \, dr \overset{*}{=}$$

$\begin{cases} u = r\cos\theta \\ v = r\sin\theta \\ r \geq 0, \; |\theta| \leq \pi \end{cases}$ $\quad du \, dv = r \, dr \, d\theta \quad$ e $\quad (u,v) \in B \iff \begin{cases} 0 \leq r \leq 1 \\ |\theta| \leq \pi \end{cases}$

$$\overset{*}{=} 9\pi \cdot \int_0^1 e^{r^2} 2r \, dr = 9\pi \cdot \left[e^{r^2}\right]_0^1 = \boxed{9\pi e}$$

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(4) (E) $y'' - 36y = \cos(6x) + e^{-6x}$

(O) $z'' - 36 \cdot z = 0$ $\qquad \lambda^2 - 36 = 0 \quad \lambda = \pm 6$:

$z(x) = A \cdot e^{6x} + B \cdot e^{-6x}$ è l'int. generale di (O).

Cerco una soluzione di (E) nella forma

$$y(x) = C \cdot \cos(6x) + D \cdot \sin(6x) + E \cdot \underbrace{x \cdot e^{-6x}}$$

moltiplico per x ovendosi risonanza tra (O) e e^{-6x}

$$y'(x) = -6C \cdot \sin(6x) + 6D \cdot \cos(6x) + E \cdot e^{-6x} - 6 \cdot E x e^{-6x}$$

$$y''(x) = -36 \cdot C \cdot \cos(6x) - 36 \cdot D \cdot \sin(6x) - 12 \cdot E e^{-6x} + 36 \cdot E \cdot x \cdot e^{-6x}$$

$$y''(x) - 36 \cdot y(x) = \cos(6x) \cdot [-72 \cdot C] + \sin(6x) \cdot [-72 \cdot D] - 12 \cdot E \cdot e^{-6x}$$

$$= \cos(6x) + e^{-6x} \iff \begin{cases} -72 \cdot C = 1 \\ -72 \cdot D = 0 \\ -12 \cdot E = 1 \end{cases} \quad \begin{cases} C = -\frac{1}{72} \\ D = 0 \\ E = -\frac{1}{12} \end{cases}$$

$$\boxed{y(x) = A \cdot e^{6x} + B \cdot e^{-6x} - \frac{1}{72} \cdot \cos(6x) - \frac{1}{12} \cdot x \cdot e^{-6x}}$$

è l'int. generale di (E).

(5) Sia $F = (P, Q, R)$: $\nabla P(x,y,z) = (2xy \cdot \alpha'(x^2+y^2), \alpha(x^2+y^2) + 2y^2 \cdot \alpha'(x^2+y^2), 0)$

$\nabla Q(x,y,z) = (\alpha(x^2+y^2) + 2x^2 \cdot \alpha'(x^2+y^2), 2xy \cdot \alpha'(x^2+y^2), 0)$

$\nabla R(x,y,z) = (0, 0, \alpha'(z))$

$$\nabla \times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (R_y - Q_z, P_z - R_x, Q_x - P_y) =$$

$$= (0, 0, \alpha(x^2+y^2) + 2x^2 \cdot \alpha'(x^2+y^2) - \alpha(x^2+y^2) - 2y^2 \cdot \alpha'(x^2+y^2))$$

$$= \boxed{(0, 0, 2 \cdot (x^2 - y^2) \cdot \alpha'(x^2+y^2))}.$$

(6) Potrei cambiare variabili $(u,v) = G(x,y)$ come in (3), rendendomi la vita più facile. Se non lo faccio:

$$\begin{cases} f_x = \left[\frac{(x+y)^2}{9} + \frac{(x-y)^2}{36} - 1\right] + \left[\frac{2}{9}(x+y) + \frac{2}{36} \cdot (x-y)\right] \cdot x \\ f_y = \left[\frac{2}{9} \cdot (x+y) + \frac{2}{36} \cdot (y-x)\right] \cdot x = 0 \end{cases} \Rightarrow x = 0 \text{ o } \frac{2}{9}(x+y) + \frac{2}{36}(y-x) = 0.$$

$\nabla f = 0 \iff$ (A) o (B) : (A) $\begin{cases} \frac{y^2}{9} + \frac{y^2}{36} - 1 = 0 \\ x = 0 \end{cases} \begin{cases} y^2 = \frac{9 \cdot 36}{45} \\ x = 0 \end{cases} \begin{cases} y = \pm \frac{6}{\sqrt{5}} \\ x = 0 \end{cases}$

(B) $\begin{cases} \frac{2}{9}(x+y) + \frac{2}{36}(y-x) = 0 \\ f_x = 0 \end{cases} \begin{cases} \left(\frac{2}{9} + \frac{2}{36}\right) y = \left(-\frac{2}{9} + \frac{2}{36}\right) x \\ f_x = 0 \end{cases} \begin{cases} 45 \cdot y = -27 \cdot x \\ f_x = 0 \end{cases}$

$$\begin{cases} y = -3/5\, x \\ \left[\frac{(2/5\, x)^2}{9} + \frac{(8/5\, x)^2}{36} - 1\right] + \left[\frac{2}{9} \cdot \left(\frac{2}{5} x\right) + \frac{2}{36}\left(\frac{8}{5} x\right)\right] x = 0 \end{cases}$$

$$\begin{cases} y = -\frac{3}{5}x \\ 1 = x^2 \cdot \left\{ \frac{4}{25 \cdot 9} + \frac{64}{25 \cdot 36} + \frac{2}{9} \cdot \frac{2}{5} + \frac{2}{36} \cdot \frac{8}{5} \right\} = x^2 \cdot \frac{4}{25 \cdot 9} \cdot \{1 + 4 + 5 + 5\} \end{cases}$$

$$= x^2 \cdot \frac{4}{5 \cdot 3}$$

$$\begin{cases} y = -\frac{3}{5}x \\ x = \pm \frac{\sqrt{15}}{2} \end{cases}$$

P.ti critici: $\boxed{\left(0, \pm \frac{6}{\sqrt{5}}\right) = E_\pm \qquad \left(\pm \frac{\sqrt{15}}{2}, \mp \frac{3\sqrt{15}}{10}\right) = F_\pm}$

$$f_{xx} = 2 \cdot \left[\frac{2}{9}(x+y) + \frac{2}{36}(x-y)\right] + \left(\frac{2}{9} + \frac{2}{36}\right)x$$

$$f_{xy} = \frac{2}{9}(x+y) + \frac{2}{36}(y-x) + \left(\frac{2}{9} - \frac{2}{36}\right)x$$

$$f_{yy} = \left(\frac{2}{9} + \frac{2}{36}\right) \cdot x$$

In $E_\pm$ abbiamo $f_{yy} = 0$ e $f_{xy} = \left(\frac{2}{9} + \frac{2}{36}\right) \cdot \left(\pm \frac{6}{\sqrt{5}}\right) \neq 0$:

$$\operatorname{Hess} f(E_\pm) = \begin{bmatrix} A & \neq 0 \\ \neq 0 & 0 \end{bmatrix}$$: $E_\pm$ p.ti di sella perché $\det(\cdot) < 0$.

In $F_\pm$ abbiamo $\frac{2}{9}(x+y) + \frac{2}{36}(y-x) = 0$:

$$\cancel{f_{xy} = \left(\frac{2}{9} - \frac{2}{36}\right) \cdot \left(\mp \frac{3\sqrt{15}}{10}\right) = \frac{27 \cdot 2}{9 \cdot 36} \cdot \left(\mp \frac{3\sqrt{15}}{10}\right) = \mp \frac{\sqrt{15}}{20}}$$

$$\cancel{f_{yy} = \frac{45 \cdot 2}{9 \cdot 36} \cdot \left(\mp \frac{3\sqrt{15}}{10}\right) = \mp \frac{\sqrt{15}}{12}}$$

$$\cancel{f_{xx} = \left\{2 \cdot \left[\frac{2}{9}\left(1 - \frac{3}{5}\right) + \frac{2}{36}\left(1 + \frac{3}{5}\right)\right] + \left(\frac{2}{9} + \frac{2}{36}\right)\right\} \cdot}$$

$$f_{xy} = \left(\frac{2}{9} - \frac{2}{36}\right) \cdot \left(\pm \frac{\sqrt{15}}{2}\right) = \frac{\cancel{2} \cdot 27}{9 \cdot 36} \cdot \left(\pm \frac{\sqrt{15}}{\cancel{2}}\right) = \pm \frac{\sqrt{15}}{12}$$

$$f_{yy} = \left(\frac{2}{9} + \frac{2}{36}\right) \cdot \left(\pm \frac{\sqrt{15}}{2}\right) = \frac{45 \cdot \cancel{2}}{9 \cdot 36} \cdot \left(\pm \frac{\sqrt{15}}{\cancel{2}}\right) = \pm \sqrt{15} \cdot \frac{5}{36}$$

$$f_{xx} = \left\{2 \cdot \left[\frac{\cancel{2}}{9}\left(1 - \frac{3}{5}\right) + \frac{\cancel{2}}{36}\left(1 + \frac{3}{5}\right)\right] + \left(\frac{\cancel{2}}{9} + \frac{\cancel{2}}{36}\right)\right\} \cdot \left(\pm \frac{\sqrt{15}}{\cancel{2}}\right)$$

$$= \pm \sqrt{15} \cdot \left\{2 \cdot \left[\frac{2}{9 \cdot 5} + \frac{8}{36 \cdot 5}\right] + \frac{45}{9 \cdot 36}\right\} = \pm \sqrt{15} \cdot \left\{\frac{8}{45} + \frac{5}{36}\right\}$$

$$= \pm \sqrt{15} \cdot \left\{\frac{32 + 25}{5 \cdot 4 \cdot 9}\right\} = \pm \sqrt{15} \cdot \frac{57}{5 \cdot 4 \cdot 9} = \pm \sqrt{15} \cdot \frac{29}{60}$$

$$\operatorname{Hess} f(F_\pm) = \pm \sqrt{15} \cdot \begin{bmatrix} \frac{29}{60} & \frac{1}{12} \\ \frac{1}{12} & 5/36 \end{bmatrix}$$: $\det[\cdot] = \frac{29}{60} \cdot \frac{5}{36} - \frac{1}{12^2} = \frac{29 - 3}{12^2 \cdot 3} > 0$

è def. pos. $\Rightarrow$

$\operatorname{Hess} f(F_+)$ def pos (F_+ min); $\operatorname{Hess} f(F_-)$ def neg. (F_- max)

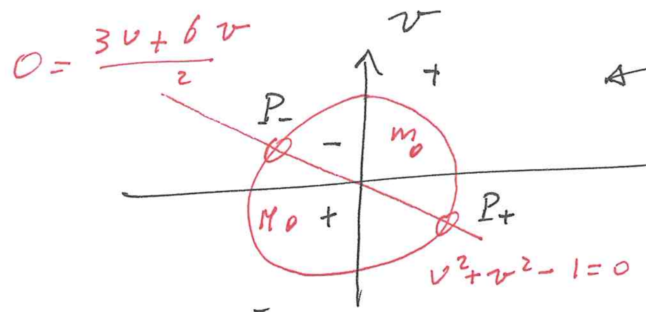
Nota. Se cambio variabili in $u = \frac{x+y}{3}$ e $v = \frac{x-y}{6}$,

cioè $x = \frac{3u+6v}{2}$, $y = \frac{3u-6v}{2}$

ho $f(x,y) = g(u,v) = (u^2+v^2-1)\cdot\frac{3u+6v}{2}$

$$\begin{cases} g_u = 2u\cdot\frac{3u+6v}{2} + (u^2+v^2-1)\cdot\frac{3}{2} \\ g_v = 2v\cdot\frac{3u+6v}{2} + (u^2+v^2-1)\cdot\frac{6}{2} \end{cases}$$

$$\nabla g = 0 \iff \left\{\begin{array}{l} 0 = 2g_u - g_v = (4u-2v)\cdot\frac{3u+6v}{2} \\ 0 = g_u = 2u\cdot\frac{3u+6v}{2} + (u^2+v^2-1)\cdot\frac{3}{2} \end{array}\right\} \iff (A) \vee (B)$$



So a priori di avere selle se $\begin{cases} 3u+6v=0 \\ u^2+v^2-1=0 \end{cases}$ e di avere almeno un p.to di min. rel. m e di MAX. rel. M approssimativamente in figura.

$(A) \iff \begin{cases} 3u+6v=0 \\ u^2+v^2=1 \end{cases}$ $\begin{cases} u=-2v \\ 4v^2+v^2=1 \end{cases}$ $\begin{cases} u = \mp 2/\sqrt{5} \\ v = \pm 1/\sqrt{5} \end{cases}$ $P_{\mp} = \mp\left(\frac{2}{\sqrt5}, -\frac{1}{\sqrt5}\right)$ p.ti di sella per g.

$(B) \iff \begin{cases} 4u-2v=0 \\ 2u\cdot\frac{3u+6v}{2} + (u^2+v^2-1)\cdot\frac{3}{2}=0 \end{cases}$ $\begin{cases} v=2u \\ u\cdot 15u + (5u^2-1)\cdot\frac{3}{2} = 0 \end{cases}$

$\begin{cases} u^2\left(15+15/2\right) = \frac{3}{2} \\ v=2u \end{cases}$ $u = \pm\sqrt{\frac{3}{45}} = \pm\frac{1}{\sqrt{15}}$

$v = \pm 2\sqrt{\frac{3}{45}} = \pm\frac{2}{\sqrt{15}}$

$m = \left(\frac{1}{\sqrt{15}}, \frac{2}{\sqrt{15}}\right)$ p.to ~~MAX~~ min rel.

$M = \left(-\frac{1}{\sqrt{15}}, \frac{-2}{\sqrt{15}}\right)$ p.to MAX. rel.

Ricambio variabili:

$E_\pm = \pm\left(\frac12\cdot\left(3\cdot\frac{2}{\sqrt5} - 6\cdot\frac{1}{\sqrt5}\right), \frac12\cdot\left(3\cdot\frac{2}{\sqrt5} + \frac{6}{\sqrt5}\right)\right) = \pm\left(0; \frac{6}{\sqrt5}\right)$ p.ti di sella per f.

$F_\pm = \pm\left(\frac12\left(\frac{3}{\sqrt{15}} + \frac{6}{\sqrt{15}}\right), \frac12\left(\frac{3}{\sqrt{15}} - \frac{6}{\sqrt{15}}\right)\right) = \ldots$ $\xrightarrow{+}$ min. rel.; $\xrightarrow{-}$ MAX. rel.