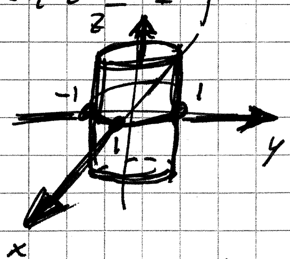
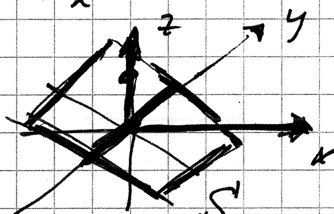
AN. MAT. II ; 10/9/2010 ; Correzioni.

$\boxed{\text{Es. 1}}$ $\Omega = \{(x,y,z) : x^2+y^2 \le 1 ;\ 0 \le x+z \le 1\}$

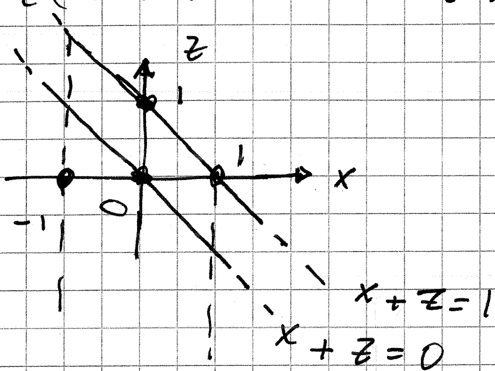


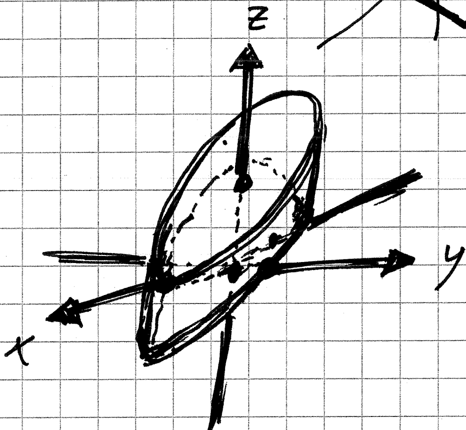
$S_1 = \{(x,y,z) : x^2+y^2=1\}$ è un cilindro

$S_2 = \{(x,y,z) : x+z=0\}$
$S_3 = \{(x,y,z) : x+z=1\}$ $\Big\}$ sono piani paralleli



S_2





Parametrizzazione del bordo

$A_1 = \{(x,y,z) : x^2+y^2=1 ;\ 0 \le x+z \le 1\}$


A_1

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = t \end{cases} \qquad \Phi_1(\theta,t) = (\cos\theta, \sin\theta, t)$$

$$0 \le \theta \le 2\pi$$

$$-\cos\theta \le t \le 1-\cos\theta$$

$$\Phi_1 : \underbrace{\{(\theta,t) : 0 \le \theta \le 2\pi ;\ -\cos\theta \le t \le 1-\cos\theta\}}_{G} \to \mathbb{R}^3$$

$A_2 = \{(x,y,z) : x^2+y^2 \le 1 ;\ x+z=0\}$


A_2

$$\begin{cases} x = u \\ y = v \\ z = -u \end{cases} \qquad \Phi_2(u,v) = (u, v, -u)$$

$$\Phi_2 : \underbrace{\{(u,v) : u^2+v^2 \le 1\}}_{B} \to \mathbb{R}^3$$

$A_3 = \{(x,y,z) : x^2+y^2 \le 1 ;\ x+z=1\}$


A_3

$$\begin{cases} x = u \\ y = v \\ z = 1-u \end{cases} \qquad \Phi_3(u,v) = (u, v, 1-u)$$

$$\Phi_3 : \underbrace{\{(u,v) : u^2+v^2 \le 1\}}_{B} \to \mathbb{R}^3$$

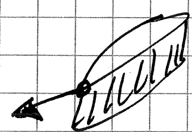
• $\partial_\theta \Phi_1(\theta,t) = (-\sin\theta, \cos\theta, 0)$

$\partial_t \Phi_1(\theta,t) = (0, 0, 1)$

$\partial_\theta \Phi_1(0,t) \times \partial_t \Phi_1(0,t) = (0,1,0)\times(0,0,1) = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= \underline{i} = (1,0,0)$

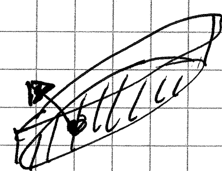
e $\Phi_1(0,t) = (1,0,t)$



Φ_1 è compatibile con ~~l'orien~~ la normale esterna

• $\partial_u \Phi_2(u,v) \times \partial_v \Phi_2(u,v) = (1,0,-1)\times(0,1,0)$

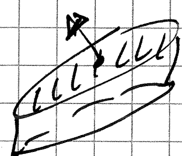
$= \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = \underline{i} + \underline{k} = (1,0,1)$



Φ_2 non è compatibile con la normale esterna

• $\partial_u \Phi_3(u,v) \times \partial_v \Phi_3(u,v) = \partial_u \Phi_2(u,v) \times \partial_v \Phi_2(u,v) = (1,0,1)$

Φ_3 è compatibile con la normale esterna.



Flusso attraverso $\partial\Omega$.

$$\iint_{\partial\Omega} F\cdot\nu \, d\sigma = \iint_{A_1} F\cdot\nu \, d\sigma + \iint_{A_2} F\cdot\nu \, d\sigma + \iint_{A_3} F\cdot\nu \, d\sigma$$

$$\iint_{A_2} F\cdot\nu \, d\sigma = -\iint_{B} (-u^2, -uv, u^2)\cdot(1,0,1)\, du\, dv$$

$$= -\iint_{B} 0 \, du\, dv = 0$$

$$\iint_{A_3} F\cdot\nu \, d\sigma = \iint_{B} \left(u(1-u);\ v(1-u);\ (1-u)^2\right)\cdot(1,0,1)\, du\, dv$$

$$= \iint_{B} (1-u)\, du\, dv$$

$$\iint_{A_1} F\cdot\nu \, d\sigma = \iint_{C'} (t\cos\theta,\ t\sin\theta, t^2)\cdot(\cos\theta, \sin\theta, 0)\, d\theta\, dt$$

usando
$$\begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \underline{i}\cos\theta + \underline{j}\sin\theta = (\cos\theta, \sin\theta, 0)$$

$$= \iint_{C'} t\, d\theta\, dt$$

cioè,
$$\boxed{\iint_{\partial\Omega} F\cdot\nu\, d\sigma = \iint_{\{(u,v):\ u^2+v^2\le 1\}} (1-v)\, du\, dv + \iint_{\{(\theta,t):\ 0\le\theta\le 2\pi,\ -\cos\theta\le t\ 1-\cos\theta\}} t\, d\theta\, dt}$$

<u>Calcolo del flusso.</u> Uso il Teorema della divergenza.

$\operatorname{div} F_0(x,y,z) = z + z + 2z = 4z$, quindi

$$\iint_{\partial\Omega} F_0\cdot\nu\, d\sigma = \iiint_{\Omega} \operatorname{div} F_0\, dx\, dy\, dz = \iiint_{\Omega} 4z\, dx\, dy\, dz$$

$$\underset{\uparrow}{=} \int_0^1 r\, dr \int_0^{2\pi} d\theta \int_{-r\cos\theta}^{1-r\cos\theta} 4z\, dz =$$

in coordinate cilindriche: $x = r\cos\theta$, $y = r\sin\theta$, $0\le r\le 1$; $0\le\theta\le 2\pi$

$$= \int_0^1 r\, dr \int_0^{2\pi} d\theta\ \left(2z^2\right)_{-r\cos\theta}^{1-r\cos\theta}$$

$$= 2\int_0^1 r\, dr \int_0^{2\pi} d\theta\ \left[(1-r\cos\theta)^2 - (-r\cos\theta)^2\right]$$

$$= 2\cdot\int_0^1 r\, dr\cdot\int_0^{2\pi} d\theta\ (1-2r\cos\theta)$$

$$= 2\cdot\int_0^1 r\, dr\cdot(2\pi - 0) = \frac{4\pi}{2} = 2\pi:$$

$$\boxed{\iint_{\partial\Omega} F\cdot\nu\, d\sigma = 2\pi}$$

$\boxed{\text{Es. 2}}$ $y'' - 4 \cdot y' = e^{2x} + e^{-2x} + 3x$

Eq. Omogenea associata: $z'' - 4z' = 0$

Eq. caratteristica: $\lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0 \iff \lambda = 0, 4$

Integrale generale dell'Omog.: $z(x) = c_1 + c_2 \cdot e^{4x}$

Poiché $\lambda = 0$ è sol. dell'Eq. caratteristica,

l'addendo $+3x$ nel termine noto è risonante.

Soluzione particolare:

$$y(x) = A e^{2x} + B e^{-2x} + C x^2 + D x$$

$$y'(x) = 2A e^{2x} - 2B e^{-2x} + 2Cx + D$$

$$y''(x) = 4A \cdot e^{2x} + 4B e^{-2x} + 2C$$

$$y'' - 4y' = -4A \cdot e^{2x} + 12B \cdot e^{-2x} - 8Cx + 2C - 4D$$

$$-4A = 1; \quad 12B = 1; \quad -8C = 3; \quad 2C - 4D = 0$$

$$\boxed{y(x) = c_1 + c_2 \cdot e^{4x} - \frac{1}{4} e^{2x} + \frac{1}{12} e^{-2x} - \frac{3}{8} x^2 - \frac{3}{16}}$$

$\boxed{\text{Es. 3}}$ $f(x,y) = (x^2 - y^2) e^{-x^2-y^2}$

$$f_x(x,y) = [2x - 2x(x^2-y^2)] e^{-x^2-y^2} = 2x(1 - x^2 + y^2) e^{-x^2-y^2}$$

$$f_y(x,y) = [-2y - 2y(x^2-y^2)] e^{-x^2-y^2} = -2y(1 + x^2 - y^2) e^{-x^2-y^2}$$

$$\nabla f(x,y) = 0 \iff \underbrace{\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \begin{cases} x = 0 \\ 1 + x^2 - y^2 = 0 \end{cases} \quad \begin{cases} 1 - x^2 + y^2 = 0 \\ y = 0 \end{cases}}_{\text{Punti caratteristici}} \quad \begin{cases} 1 - x^2 + y^2 = 0 \\ 1 + x^2 - y^2 = 0 \end{cases}$$

$$(0,0) \qquad (0, \pm 1) \qquad (\pm 1; 0) \qquad \text{imposs.}$$

$$f_{xx}(x,y) = 2 \cdot [(1 - x^2 + y^2) - 2x^2 - 2x^2(1 - x^2 + y^2)] \cdot e^{-x^2-y^2}$$

$$f_{xy}(x,y) = 2x \cdot [2y - 2y(1 - x^2 + y^2)] \cdot e^{-x^2-y^2}$$

$$= 4xy\,[x^2 - y^2] \cdot e^{-x^2-y^2}$$

$$f_{yy}(x,y) = -2 \cdot [(1 + x^2 - y^2) - 2y^2 - 2y^2(1 + x^2 - y^2)] e^{-x^2-y^2}$$

$\text{Hess} f(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ p.to di sella (Autoval. ± 2)

$\text{Hess} f(\pm 1, 0) = 2 \cdot e^{-1} \cdot \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$ p.ti di MAX. rel.

$\text{Hess} f(0, \pm 1) = 2 \cdot e^{-1} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ p.ti di min. rel.

$f(2,0) = 4 \cdot e^{-4}$; $\nabla f(2,0) = (-12 \cdot e^{-4}; 0)$

$$\text{Hess} f(2,0) = e^{-4} \cdot 2 \cdot \begin{bmatrix} 1-\cancel{4}-8-8(-3) & 0 \\ 0 & (-1)[5] \end{bmatrix}$$

$$= e^{-4} \cdot 2 \cdot \begin{bmatrix} 13 & 0 \\ 0 & -5 \end{bmatrix}$$

$$f(x,y) = 4 \cdot e^{-4} - 12 \cdot e^{-4}(x-2) + e^{-4} \cdot [14(x-2)^2 - 5y^2] + o((x-2)^2 + y^2)$$
$(x-2, y) \to (2,0)$ (F. Taylor II ordine).

Piano tangente al grafico di f in $(2,0)$: $z = 4 \cdot e^{-4} - 12 \cdot e^{-4}(x-2)$

Spazio tangente al grafico di f in $(2,0)$: $z = -12 e^{-4} \cdot x$

Una base: $\{(1, 0, -12 \cdot e^{-4}), (0, 1, 0)\}$.

$\boxed{\text{Es. 4}}$

$$J h(x_0, y_0) = \begin{bmatrix} f'(x_0 y_0) x_0 & f'(x_0 y_0) \cdot y_0 \\ f'(x_0) \cdot f(y_0) & f(x_0) f'(y_0) \\ f(y_0) - y_0 f'(x_0) & x_0 f'(y_0) - f(x_0) \end{bmatrix}$$