Prova scritta di Analisi Matematica II (17/6/2013)

Nome..Cognome.. Matricola................
Prova orale: inizio appello/fine appello (cancellare se non interessa),
non nella mezza giornata di..

(1) [14 pti] Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \left\{(x,y,z):\ e^{-z^2} \leq \frac{x^2}{4} + \frac{y^2}{9} \leq e^{z^2};\ 0 \leq z \leq 1\right\}$.

(1.1) Fare un disegno *qualitativo* di Ω.

(1.2) Parametrizzare $\partial\Omega$ e dire se le parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$. (Nella formula devono apparire, magari iterati, solo integrali di una variabile).

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x,y,z) = (-yz, xz, z^2)$.

(1.5) . Sia $\Sigma = \left\{(x,y,z):\ e^{-z^2} = \frac{x^2}{4} + \frac{y^2}{9};\ 0 \leq z \leq 1\right\}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la normale ν a Σ (ν essendo la normale di cui al punto (1.2)).

(1.6) Calcolare $\iint_\Sigma (\nabla \times F) \cdot \mu d\sigma$, con la stessa F di (1.4).

(2) [2 pti] Dire per quale valore del parametro $\alpha \in \mathbb{R}$ il campo $F_\alpha : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ è chiuso, dove

$$F_\alpha(x,y) = \left(e^{xy} + xye^{xy} + y^4, x^2e^{xy} + \alpha y^3\right).$$

Dire per quali di questi F è anche esatto.

(3) [5 pti] Sia $A = \left\{(x,y) : \frac{(x+y)^2}{4} + \frac{(x-y)^2}{9} \leq 1\right\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A \frac{(x+y)^2}{\frac{(x+y)^2}{4} + \frac{(x-y)^2}{9} + 1} dxdy.$$

(4) **[3 pti]** Trovare l'integrale generale di

$$y'' - 4y = \cos(2x) + e^{2x}$$

(5) **[2 pti]** Siano $\alpha, \beta \in C^1(\mathbb{R}^2, \mathbb{R})$ e $h \in C^1(\mathbb{R}^2, \mathbb{R})$ e si definisca $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x, y) = h(\alpha(x, y) + \beta(x, y), \alpha(x, y) \cdot \beta(x, y)).$$

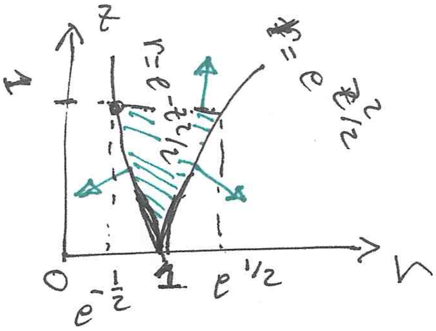
Calcolare il gradiente di f in (x, y).

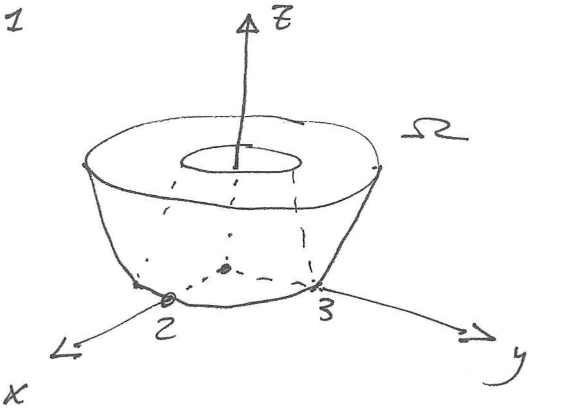
(6) **[4 pti]** Classificare i punti critici di $f(x, y) = y(y-1)(2x+y-2)(2x-y+2)$.

(1) (1.1) $\begin{cases} \frac{x}{2} = r\cos\theta & -\pi \le \theta \le \pi \\ \frac{y}{3} = r\sin\theta & 0 \le r \end{cases}$

$e^{-z^2} \le r^2 \le e^{z^2}$, cioè $e^{-z^2/2} \le r \le e^{z^2/2}$

$0 \le z \le 1$





(1.2)

$\partial\Omega = \Sigma_+ \cup \Sigma_a \cup \Sigma_b$

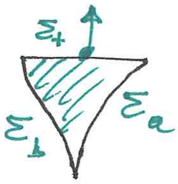
$\Sigma_+ = \left\{(x,y,1):\ e^{-1} \le \frac{x^2}{4} + \frac{y^2}{9} \le e\right\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_+ = \left\{(x,y):\ e^{-1} \le \frac{x^2}{4} + \frac{y^2}{9} \le e\right\} \xrightarrow{\Phi_+} \mathbb{R}^3$

$\Phi_+(x,y) = (x,y,1) \qquad \Phi_+(A_+) = \Sigma_+$

$\partial_x \Phi_+ \times \partial_y \Phi_+ = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$

compatibile



$\Sigma_a = \left\{(x,y,z):\ \frac{x^2}{4} + \frac{y^2}{9} = e^{z^2}\right\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_a = \{(z,\theta):\ 0 \le z \le 1;\ -\pi \le \theta \le \pi\} \xrightarrow{\Phi_a} \mathbb{R}^3$

$\Phi_a(z,\theta) = \left(2e^{z^2/2}\cos\theta;\ 3e^{z^2/2}\sin\theta;\ z\right) \qquad \Phi_a(A_a) = \Sigma_a$

$\partial_z \Phi_a \times \partial_\theta \Phi_a = \begin{vmatrix} i & j & k \\ 2z e^{z^2/2}\cos\theta & 3z e^{z^2/2}\sin\theta & 1 \\ -2e^{z^2/2}\sin\theta & 3e^{z^2/2}\cos\theta & 0 \end{vmatrix} = \left(-3e^{z^2/2}\cos\theta,\ -2e^{z^2/2}\sin\theta,\ 6z e^{z^2}\right)$

positivo

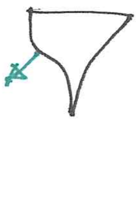
non compatibile

$\Sigma_b = \left\{(x,y,z):\ \frac{x^2}{4} + \frac{y^2}{9} = e^{-z^2}\right\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_b = A_a \xrightarrow{\Phi_b} \mathbb{R}^3;\quad \Phi_b(z,\theta) = \left(2e^{-z^2/2}\cos\theta,\ 3e^{-z^2/2}\sin\theta,\ z\right);\ \Phi_b(A_b) = \Sigma_b$

$\partial_z \Phi_b \times \partial_\theta \Phi_b = \begin{vmatrix} i & j & k \\ -2z e^{-z^2/2}\cos\theta & -3z e^{-z^2/2}\sin\theta & 1 \\ -2e^{-z^2/2}\sin\theta & 3e^{-z^2/2}\cos\theta & 0 \end{vmatrix} = \left(-3e^{-z^2/2}\cos\theta,\ -2e^{-z^2/2}\sin\theta,\ -6z e^{-z^2}\right)$

negativo



compatibile

(1.3) Uso T. div.: $\iint_{\partial\Omega} F\cdot\nu\, d\sigma = \iiint_{\Omega} \operatorname{div} F(x,y,z)\, dx\, dy\, dz$

$dx\,dy\,dz = 6r\,dr\,d\theta\,dz$

$$= \int_{-\pi}^{\pi} d\theta \int_0^1 dz \int_{e^{-z^2/2}}^{e^{z^2/2}} 6r\,dr \cdot \operatorname{div} F(2r\cos\theta, 3r\sin\theta, z).$$

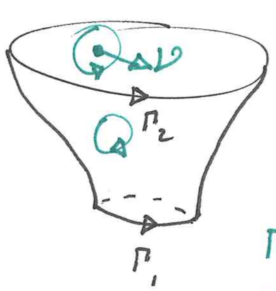
(1.4) $\operatorname{div} F(x,y,z) = 0 + 0 + 2z$:

$$\iint_{\partial\Omega} F\cdot\nu\, d\sigma = \int_{-\pi}^{\pi} d\theta \int_0^1 dz \int_{e^{-z^2/2}}^{e^{z^2/2}} 6r\,dr \cdot 2z$$

$$= 2\pi\cdot 6\cdot 2\cdot \int_0^1 z\,dz \cdot \left(\frac{r^2}{2}\right)\Big|_{e^{-z^2/2}}^{e^{z^2/2}} = 2\pi\cdot 6\cdot \int_0^1 z\,(e^{z^2} - e^{-z^2})\,dz$$

$$= 6\pi\cdot \left(e^{z^2} + e^{-z^2}\right)\Big|_{z=0}^{z=1} = 6\pi\cdot(e + e^{-1} - 2)$$

(1.5) $\Sigma = \Sigma_b$



$\Gamma_1(\theta) = (2\cos\theta;\ 3\sin\theta,\ 0)$
$|\theta| \le \pi$
$\Gamma_2(\theta) = (2e^{-1/2}\cos\theta;\ 3e^{-1/2}\sin\theta,\ 0)$
Γ_1: non compatibile
Γ_2: compatibile

(1.6) Uso T. Stokes? $\iint_{\Sigma} (\nabla\times F)\cdot\nu\, d\sigma$

$$\nabla\times F(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = (-x, -y, 2z) =$$

$$= (-2e^{-z^2/2}\cos\theta,\ -3e^{-z^2/2}\sin\theta,\ 2z)$$

NO: Uso i cambi per $\Sigma_b = \Sigma$

$$\iint_{\Sigma} (\nabla\times F)\cdot\nu\, d\sigma = + \int_0^1 dz \int_{-\pi}^{\pi} d\theta\, (-2e^{-z^2/2}\cos\theta,\ -3e^{-z^2/2}\sin\theta,\ 2z)\cdot(-3e^{-z^2/2}\cos\theta,\ -2e^{-z^2/2}\sin\theta,\ -6z\,e^{-z^2})$$

per compatibile

$$= \int_0^1 dz \int_{-\pi}^{\pi} d\theta\, (6e^{-z^2} + 12z^2 e^{-z^2})$$

$$= 12\cdot\pi\cdot \int_0^1 (e^{-z^2} + 2z^2\cdot e^{-z^2})\,dz$$ ⚠ più in là non posso andare

(2) $F=(P,Q)$: $P_y = x e^{xy} + x e^{xy} + x^2 y e^{xy} + 4y^3$

$Q_x = 2x e^{xy} + x^2 y e^{xy} + 0$

$\forall \alpha \in \mathbb{R}$: $P_y \neq Q_x$. Mai chiuso, mai esatto.

(3)
$$\begin{cases} \dfrac{x+y}{2} = r\cos\theta = u \\ \dfrac{x-y}{3} = r\sin\theta = v \end{cases}$$

(due cambi di variabili)

$$J\begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}; \quad du\,dv = \frac{dx\,dy}{3}$$

$(x,y) \in A \iff u^2 + v^2 \leq 1$

$$\iint_A \frac{(x+y)^2}{\frac{(x+y)^2}{4} + \frac{(x-y)^2}{9} + 1} \,dx\,dy = \iint_{\{(u,v):\ u^2+v^2 \leq 1\}} \frac{(2u)^2}{u^2+v^2+1} \cdot 3\,du\,dv$$

$$= 12 \int_0^1 r\,dr \int_{-\pi}^{\pi} d\theta \cdot \frac{r^2\cos^2\theta}{r^2+1}$$

$$= 6 \cdot \int_{-\pi}^{\pi} \cos^2\theta\,d\theta \cdot \int_0^1 2r \cdot \frac{r^2}{r^2+1}\,dr$$

$$= 6 \cdot \pi \cdot \int_0^1 \frac{t}{t+1}\,dt = 6\pi \cdot \int_0^1 \left(1 - \frac{1}{t+1}\right) dt$$

$$= 6\pi \cdot \left[1 - \left(\log|t+1|\right)_0^1\right] = 6\pi \cdot (1 - \log 2)$$

(4) $y'' - 4y = \cos(2x) + e^{2x}$

$z'' - 4z = 0$

$\lambda^2 - 4 = 0$

$\lambda = \pm 2 \qquad z(x) = A \cdot e^{2x} + B e^{-2x}$

$y(x) \overset{?}{=} C \cdot \cos(2x) + D \cdot \sin(2x) + E \cdot x e^{2x}$

$y''(x) = -4 \cdot C \cdot \cos(2x) - 4D \cdot \sin(2x) + E \cdot 4e^{2x}(1+x)$

$y''(x) - 4 \cdot y(x) = -8 \cdot C \cdot \cos(2x) - 8 \cdot D \cdot \sin(2x) + E \cdot 4 e^{2x}$

$(x e^{2x})' = e^{2x} + 2x e^{2x}$

$(x e^{2x})'' = 4 e^{2x} + 4x e^{2x}$

$C = -\frac{1}{8}$; $D = 0$; $E = \frac{1}{4}$

$$\boxed{y(x) = -\frac{1}{8}\cos(2x) + \frac{1}{4} x e^{2x} + A \cdot e^{2x} + B \cdot e^{-2x}}$$

(5) $\partial_x f(x,y) = \partial_v h(v,w)\,[\partial_x \alpha(x,y) + \partial_x \beta(x,y)]$

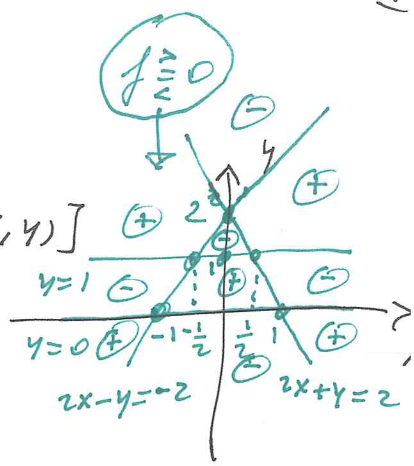
$+\, \partial_w h(v,w)\cdot[\partial_x \alpha(x,y)\cdot \beta(x,y) + \alpha(x,y)\cdot \partial_x \beta(x,y)]$

$$\boxed{\begin{aligned} h &= h(v,w) \\ v &= \alpha(x,y) + \beta(x,y) \\ w &= \alpha(x,y)\cdot \beta(x,y) \end{aligned}}$$

$\partial_y f(x,y) = \partial_v h(v,w)\cdot[\partial_y \alpha(x,y) + \partial_y \beta(x,y)]$

$+\, \partial_w h(v,w)\cdot[\partial_y \alpha(x,y)\cdot \beta(x,y) + \alpha(x,y)\cdot \partial_y \beta(x,y)]$

(6) $f(x,y) = (y^2-y)\cdot[4x^2-(y-2)^2]$



$$\begin{cases} \partial_x f(x,y) = (y^2-y)\cdot 8x = 0 \\ \partial_y f(x,y) = (2y-1)\cdot[4x^2-(y-2)^2] + (y^2-y)\cdot[-2(y-2)] = 0 \end{cases}$$

$$\begin{cases} y^2-y=0 \\ (2y-1)[4x^2-(y-2)^2]=0 \end{cases} \quad \text{oppure} \quad B\begin{cases} x=0 \\ -(y-2)^2(2y-1) - 2(y-2)(y^2-y)=0 \end{cases}$$

$$A1\begin{cases} y=0 \\ 4x^2-4=0 \end{cases} \quad \text{oppure} \quad A2\begin{cases} y=1 \\ 4x^2-1=0 \end{cases}$$

$$\underbrace{(\pm 1, 0) \qquad (\pm \tfrac{1}{2}, 1)}_{\text{punti di sella, come previsto}}$$

$$B\begin{cases} x=0 \\ (y-2)\cdot[-(y-2)(2y-1)-2(y^2-y)]=0 \end{cases}$$

$$\begin{cases} x=0 \\ (y-2)\cdot(-4y^2+7y-2)=0 \end{cases}$$

$$\begin{cases} x=0 \\ (y-2)(4y^2-7y+2)=0 \end{cases}$$

A PRIORI so che

- $(0,2)$; $(\pm\frac{1}{2},1)$; $(\pm 1,0)$ sono punti di sella
- ho un punto di min. rel. in triangolo $(0,2)$, $(-\frac{1}{2},1)$, $(\frac{1}{2},1)$
- ho un punto di max. rel. in trapezio $(-\frac{1}{2},1)$, $(\frac{1}{2},1)$, $(-1,0)$, $(1,0)$

Almeno 7 pti critici

$(0,2) \leftarrow$ punto di sella, previsto

$\left(0, \frac{7+\sqrt{17}}{8}\right) \leftarrow$ punto di min. rel.

$\left(0, \frac{7-\sqrt{17}}{8}\right) \leftarrow$ punto di MAX rel.

$$\boxed{\begin{aligned} &(\pm 1,0);\ (\pm\tfrac{1}{2},1);\ (0,2) \to \text{p.ti sella} \\ &\left(0, \tfrac{7+\sqrt{17}}{8}\right) \to \text{p.to min. rel.} \\ &\left(0, \tfrac{7-\sqrt{17}}{8}\right) \to \text{p.to MAX rel.} \end{aligned}}$$

AMLB - 17/06/2013

(1) $(z+2i)^2 + 12\cdot(z+2i) + 35 = 0$

$$z+2i = -6 \pm \sqrt{36-35} = -5, -7$$

$$\boxed{\begin{cases} z = -5-2i \\ z = -7-2i \end{cases}}$$

(2) $$\lim_{x \to 1^+} \frac{\arctan(x^\gamma)}{x^{2\gamma}+x^{3\gamma}} = \frac{\arctan(1)}{2} > 0$$

L'integrale I converge $\iff 3\gamma > 1 \iff \gamma > 1/3$

$$I = \int_1^{+\infty} \frac{\arctan(x^\gamma)}{x^{2\gamma}+x^{3\gamma}}\,dx \qquad f(x) = \frac{\arctan(x^\gamma)}{x^{2\gamma}+x^{3\gamma}} \underset{x\to+\infty}{\sim} \frac{\arctan(\infty)}{x^{3\gamma}} = \frac{\pi/2}{x^{3\gamma}}$$

(3) $\Omega = \{(x,y,z) : 0 \le x \le 1;\ 0 \le y \le 1;\ xy \le z \le xy+1\}$.

$$\iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_0^2 \left\{ \iint_{\{(x,y) : z-1 \le xy \le z\}} f(x,y,z)\,dx\,dy \right\} dz$$