# Prova scritta di Analisi Matematica L-B

22 luglio 2011

**Nome**............................................**Cognome**................................................ **Matricola**..................

**Prova orale: non** nel giorno..................................

**(1) [4 pti]** Sia $A = \left\{(x,y):\ \frac{x^2}{4} + \frac{y^2}{9} \leq 1,\ x \geq 0\right\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A \frac{x+1}{1+\frac{x^2}{4}+\frac{y^2}{9}}\,dxdy.$$

**(2) [8 pti]** Classificare i punti critici di $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \left(\frac{x^2}{4} - \frac{y^2}{9} - 1\right)\left(\frac{x^2}{4} - 4\right) + 1$.

**(3) [4 pti]** Trovare l'integrale generale di $\dot{x} + \cancel{2}\cos(t)\cdot x = te^{-\sin(t)}$.

**(4) [3 pti]** Siano $\varphi \in C^1(\mathbb{R},\mathbb{R})$, $\gamma \in C^1(\mathbb{R},\mathbb{R}^3)$ e $f \in C^1(\mathbb{R}^3,\mathbb{R})$. Si definisca $F : \mathbb{R}^3 \to \mathbb{R}^3$,

$$F(x,y,z) = (x\varphi(z), y\varphi(z), f(x,y,z)).$$

Calcolare, per $t \in \mathbb{R}$,

$$\frac{d}{dt}(F \circ \gamma)(t).$$

**(5) [5 pti]** Siano $\Omega \subset \mathbb{R}^3$ l'insieme

$$\Omega = \left\{ (x,y,z) : \ 1 \leq \frac{x^2}{4} + \frac{y^2}{9} \leq 4 - z^2, \ z \geq 0 \right\}.$$

e $f \in C(\Omega,\mathbb{R})$ continua.

Trovare $A \subset \mathbb{R}^2$ e, per $(x,y) \in \mathbb{R}^2$, trovare $\alpha(x,y), \beta(x,y) \in \mathbb{R}$, tali che

$$\iiint_\Omega f(x,y,z)dxdydz = \iint_A \left[ \int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)dz \right] dxdy$$

**(6) [3 pti]** Trovare le soluzioni in $\mathbb{C}$ dell'equazione

$$z^4 - 3iz^2 - 2 = 0$$

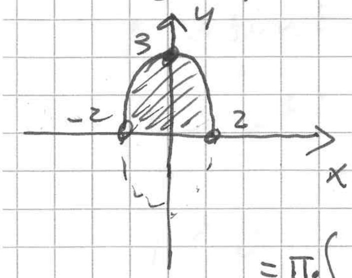
e calcolarne la parte reale.

**(7) [3 pti]** Trovare i valori di $\gamma \geq 0$ tali che converga l'integrale generalizzato:

$$\int_0^\infty \frac{1-e^{-x}}{(x^\gamma + x^{2\gamma})^2}.$$

① Pongo $\begin{cases} x/2 = r\cos\theta \\ y/3 = r\sin\theta \end{cases}$ $\left| \begin{matrix} r \geq 0 \\ |\theta| \leq \pi \end{matrix} \right|$ $dx\,dy = 6r\,dr\,d\theta$

$(x,y) \in A \iff r^2 \leq 1;\ \cos\theta \geq 0 \iff 0 \leq r \leq 1$ e $|\theta| \leq \pi/2$



$$\iint_A \frac{x+1}{1+\frac{x^2}{4}+\frac{y^2}{9}}\,dx\,dy = \int_0^1 r\,dr \int_{-\pi/2}^{\pi/2} d\theta\ \frac{1+2r\cos\theta}{1+r^2}$$

$$= \pi \cdot \int_0^1 \frac{r\,dr}{1+r^2} + \int_{-\pi/2}^{\pi/2} \cos\theta \cdot d\theta \cdot 2 \cdot \int_0^1 \frac{r^2\,dr}{1+r^2}$$

$$= \frac{\pi}{2} \cdot \log(1+r^2)\Big|_0^1 + \sin\theta\Big|_{-\pi/2}^{\pi/2} \cdot 2 \cdot \int_0^1 \left(1 - \frac{1}{1+r^2}\right) dr$$

$$= \pi/2 \cdot \log 2 + 4 \cdot \left(1 - \operatorname{arctg}(r)\Big|_0^1\right) = \pi/2 \cdot \log 2 + 4\left(1 - \pi/4\right)$$

② $f_x = \frac{x}{2} \cdot \left(\frac{x^2}{4} - 4\right) + \left(\frac{x^2}{4} - \frac{y^2}{9} - 1\right) \cdot \frac{x}{2} = \frac{x}{2} \cdot \left(\frac{x^2}{2} - \frac{y^2}{9} - 5\right)$

$f_y = -\frac{2}{9} y \cdot \left(\frac{x^2}{4} - 4\right) = 0 \iff y = 0$ o $\frac{x^2}{4} - 4 = 0$

$\nabla f(x,y) = 0 \iff$ (i) $\begin{cases} y = 0 \\ \frac{x}{2} \cdot \left[\left(\frac{x^2}{4} - 4\right) + \left(\frac{x^2}{4} - 1\right)\right] = \frac{x}{2} \cdot \left(\frac{x^2}{2} - 5\right) = 0 \end{cases}$

o (ii) $\begin{cases} x^2/4 - 4 = 0 \\ x/2 \cdot \left(\frac{x^2}{4} - \frac{y^2}{9} - 1\right) = 0 \end{cases}$

(i) $\begin{cases} y = 0 \\ x = 0 \end{cases}$ o $\begin{cases} y = 0 \\ x = \pm\sqrt{10} \end{cases}$ (ii) $\begin{cases} x = 4 \\ y = \pm 3\sqrt{3} \end{cases}$ o $\begin{cases} x = -4 \\ y = \pm 3\sqrt{3} \end{cases}$

Punti critici: $(0,0)$, $(\pm\sqrt{10}, 0)$, $(4, \pm 3\sqrt{3})$, $(-4, \pm 3\sqrt{3})$

$f_{xy} = -\frac{xy}{9}$; $f_{yy} = -\frac{2}{9}\left(\frac{x^2}{4} - 4\right)$; $f_{xx} = \frac{1}{2}\left(\frac{x^2}{2} - \frac{y^2}{9} - 5\right) + \frac{x^2}{2}$

$\text{Hess} f(0,0) = \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix}$: selle; $\text{Hess} f(\pm 4, \pm 3\sqrt{3}) = \begin{pmatrix} \ast & \ast \\ \ast & 0 \end{pmatrix}$ selle

$\text{Hess} f(\sqrt{10}, 0) = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$ e $\text{Hess} f(-\sqrt{10}, 0) = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$

punti di min. relativo;

Selle: $(0,0)$; $(\pm 4, \pm 3\sqrt{3})$ e punti min. rel. $(\pm\sqrt{10}, 0)$

(3) Eq. Diff. Lin. del I ordine, del tipo

$$\dot{x} + a(t)\, x = f(t).$$

Cerco una primitiva di $a(t) = \cos(t)$: $\int a(t)\,dt = \sin(t)$

$$\dot{x}\, e^{\sin t} + \cos t \cdot e^{\sin t} \cdot x = t$$

$$(x \cdot e^{\sin t})' = t \qquad x \cdot e^{\sin t} = k + \frac{t^2}{2} \quad (k \in \mathbb{R})$$

$$\boxed{x(t) = \left(k + \frac{t^2}{2}\right) \cdot e^{-\sin(t)}}, \quad x \in C^1(\mathbb{R}, \mathbb{R})$$

(4) Sia $\gamma = (\gamma_1, \gamma_2, \gamma_3)$. Allora

$$F(\gamma(t)) = (\gamma_1(t)\,\varphi(\gamma_3(t)),\ \gamma_2(t)\,\varphi(\gamma_3(t)),\ f(\gamma_1(t), \gamma_2(t), \gamma_3(t)))$$

$$\frac{d}{dt}(F \circ \gamma)(t) = JF(\gamma(t)) \cdot \dot{\gamma}(t) \text{ e}$$

$$JF(x,y,z) = \begin{bmatrix} \varphi(z) & 0 & x\,\varphi'(z) \\ 0 & \varphi(z) & y\,\varphi'(z) \\ \partial_x f(x,y,z) & \partial_y f(x,y,z) & \partial_z f(x,y,z) \end{bmatrix}$$

$$\Rightarrow \frac{d}{dt}(F \circ \gamma)(t) = \begin{bmatrix} \varphi(\gamma_3(t)) & 0 & \gamma_1(t) \cdot \varphi'(\gamma(t)) \\ 0 & \varphi(\gamma_3(t)) & \gamma_2(t) \cdot \varphi'(\gamma(t)) \\ \partial_x f(\gamma(t)) & \partial_y f(\gamma(t)) & \partial_z f(\gamma(t)) \end{bmatrix} \begin{pmatrix} \dot{\gamma}_1(t) \\ \dot{\gamma}_2(t) \\ \dot{\gamma}_3(t) \end{pmatrix}$$

$$= \begin{bmatrix} \varphi(\gamma_3(t))\,\dot{\gamma}_1(t) + \gamma_1(t)\,\varphi'(\gamma(t))\,\dot{\gamma}_3(t) \\ \varphi(\gamma_3(t))\,\dot{\gamma}_2(t) + \gamma_2(t)\,\varphi'(\gamma(t))\,\dot{\gamma}_3(t) \\ \partial_x f(\gamma(t))\,\dot{\gamma}_1(t) + \partial_y f(\gamma(t)) \cdot \dot{\gamma}_2(t) + \partial_z f(\gamma(t))\,\dot{\gamma}_3(t) \end{bmatrix}$$

(5) Affinché esistano $z$: $1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 4 - z^2$ devo avere che $1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 4$ (poiché $4 - z^2 \le 4$)

$A = \left\{(x,y) : 1 \le \frac{x^2}{4} + \frac{y^2}{9} \le 4\right\}$ e $\forall (x,y) \in A$:

$$(x,y,z) \in \Omega \iff z \ge 0 \text{ e } z^2 \le 4 - x^2/4 - y^2/9$$

$$\iff \alpha(x,y) = 0 \le z \le \sqrt{4 - x^2/4 - y^2/9} = \beta(x,y)$$

(6) Pongo $w = z^2$. $\quad w^2 - 3iw - 2 = 0$

$$\Delta = b^2 - 4ac = (-3i)^2 - 4\cdot(-2) = -9 + 8 = -1 = i^2$$

$$w = \frac{3i \pm i}{2} = 2i,\ i$$

$$z^2 = 2i = 2\cdot e^{i\pi/2} \quad \Big| \quad z = \pm\sqrt{2}\cdot e^{i\pi/4} = \pm\sqrt{2}\cdot\left[\cos(\pi/4) + i\sin\left(\tfrac{\pi}{4}\right)\right]$$

o

$$z^2 = i = e^{i\pi/2} \quad \Big| \quad z = \pm e^{i\pi/4} = \pm\left[\cos(\pi/4) + i\sin(\pi/4)\right]$$

$$z = \pm\sqrt{2}\cdot\left(1/\sqrt{2} + i/\sqrt{2}\right) = \pm(1+i)$$

o $z = \pm\left(1/\sqrt{2} + i/\sqrt{2}\right)$

(7) Per $x \to 0$, $\quad 1 - e^{-x} = 1 - (1 - x + o(x)) = x + o(x)$

$$\Rightarrow \frac{1 - e^{-x}}{(x^\gamma + x^{2\gamma})^2} = \frac{x + o(x)}{x^{2\gamma} + o(x^{2\gamma})} = \frac{1}{x^{2\gamma - 1}}\cdot(1 + o(1))$$

e l'integrale converge in $(0,1] \Leftrightarrow 2\gamma - 1 < 1 \Leftrightarrow \gamma < 1$

Per $x \to \infty$, $\quad 1 - e^{-x} \to 1$

$$\Rightarrow \frac{1 - e^{-x}}{(x^\gamma + x^{2\gamma})^2} = \frac{1 + o(1)}{x^{4\gamma} + o(x^{4\gamma})} = \frac{1}{x^{4\gamma}}\cdot(1 + o(1))$$

e l'integrale converge in $[1, \infty) \Leftrightarrow 4\gamma > 1 \Leftrightarrow \gamma > \frac{1}{4}$

L'integrale converge $\Leftrightarrow \frac{1}{4} < \gamma < 1$.