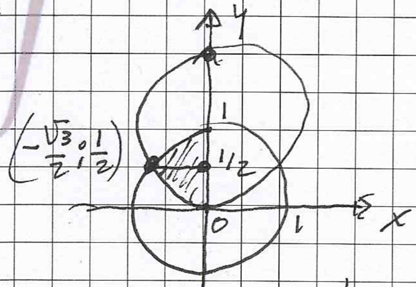
XIX CONGRESSO DELL'UNIONE MATEMATICA ITALIANA

BOLOGNA 12 - 17 SETTEMBRE 2011

① $A = \{(x,y):\ x^2+y^2<1;\ \underbrace{x^2+y^2<2y}_{(II)};\ x<0\}$



$x^2+(y-1)^2<1$

$I = \iint_A x\cdot\sqrt{x^2+y^2}\,dx\,dy =$

$= \int_0^{1/2} dy \int_{-\sqrt{2y-y^2}}^{0} dx\cdot x\sqrt{x^2+y^2} + \int_{1/2}^{1} dy \int_{-\sqrt{1-y^2}}^{0} dx\cdot x\cdot\sqrt{x^2+y^2}$

$= \int_0^{1/2} dy\cdot\left[\frac{(x^2+y^2)^{3/2}}{3}\right]_{x=-\sqrt{2y-y^2}}^{x=0} + \int_{1/2}^{1} dy\left[\frac{(x^2+y^2)^{3/2}}{3}\right]_{x=-\sqrt{1-y^2}}^{x=0}$

$=\frac13\int_0^{1/2}\left[y^3-(2y)^{3/2}\right]dy+\frac13\int_{1/2}^{1}\left[y^3-1\right]dy = \frac13\int_0^1 y^3\,dy-\frac13\int_0^{1/2}(2y)^{3/2}dy-\frac16$

$=\frac13\left(\frac{y^4}{4}\right)_0^1-\frac13\left(\frac{(2y)^{5/2}}{5}\right)_0^{1/2}-\frac16=\frac1{12}-\frac1{15}-\frac16=\frac{5-4-10}{60}=-\frac{3}{20}$

In coordinate polari $\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}$ $|\theta|\le\pi$, $r\ge0$:

$I=\int_{\pi/2}^{\pi/2+\pi/3}d\theta\int_0^1 r\,dr\cdot r\cos\theta\cdot r+\int_{\pi/2+\pi/3}^{\pi}d\theta\int_0^{2\sin\theta}r\,dr\cdot r\cos\theta\cdot r$

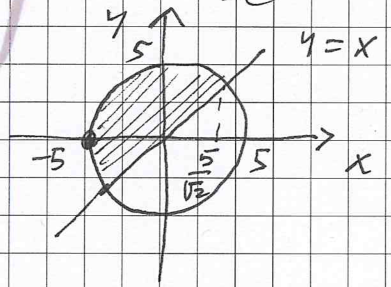
$=(\sin\theta)_{\pi/2}^{\pi/2+\pi/3}\cdot\left(\frac{r^4}{4}\right)_0^1+\int_{\pi/2+\pi/3}^{\pi}\cos\theta\cdot\left(\frac{r^4}{4}\right)_0^{2\sin\theta}$

$=\left(\frac12-1\right)\cdot\frac14+\frac{2^4}{4}\cdot\int_{\pi/2+\pi/3}^{\pi}\cos\theta\cdot\sin^4\theta\,d\theta$

$=-\frac18+4\cdot\left(\frac{\sin^5\theta}{5}\right)_{\pi/2+\pi/3}^{\pi}=-\frac18-\frac45\cdot\left(\frac12\right)^5=-\frac18-\frac1{40}=-$

(2) $f(x,y) = x + y^2$

$M = \{(x,y) : x^2+y^2 \leq 25 ; \; x \leq y\}$



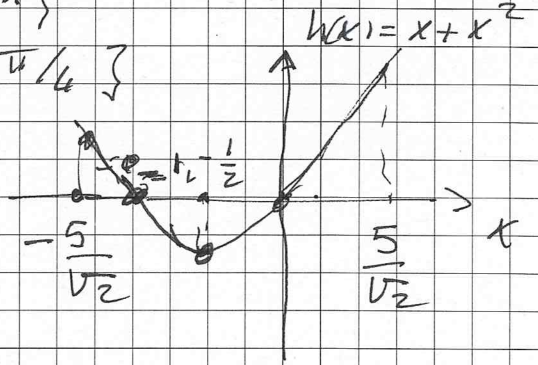
$\partial M = K_1 \cup K_2$

$K_1 = \{(x,x) : |x| \leq 5/\sqrt{2}\}$

$= \{(x,y) : x = y \text{ e } x^2+y^2 \leq 25\}$

e $K_2 = \{(x,y) : x^2+y^2 = 25 \text{ e } y \geq x\}$

$= \{(5\cos\theta, 5\sin\theta) : \pi/4 \leq \theta \leq \pi + \pi/4\}$



Su K_1, $h(x) := f(x,x) = \cancel{x} + x^2$

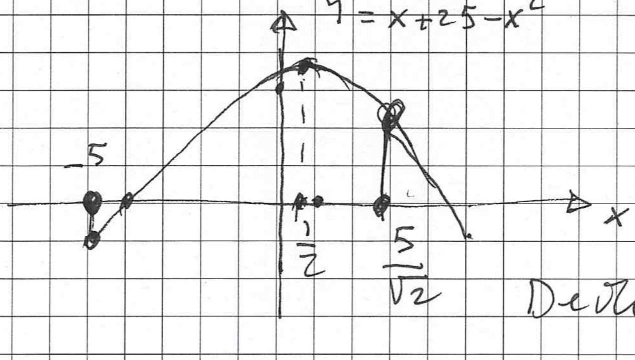
min di f su K_1 è

$h(-\frac{1}{2}) = f(-\frac{1}{2}, -\frac{1}{2}) = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

MAX di f su K_1 è $h(5/\sqrt{2}) = \frac{5}{\sqrt{2}} + \frac{25}{2}$

Su K_2, $y^2 = 25 - x^2$, quindi

$g(x) = f(x,y) = x + 25 - x^2$ con $-5 \leq x \leq 5/\sqrt{2}$



MAX di f su K_2 è $g(\frac{1}{2}) = 25 + \frac{1}{2} - \frac{1}{4}$

$= 25 + \frac{1}{4}$

e min di f su K_2 è $g(-5) = -5$

Deduco che $\max_{\partial M} f = 25 + \frac{1}{4}$; $\min_{\partial M} f = -5$

Con i moltiplicatori. Sia $a(x,y) = x - y$.

$\nabla f - \lambda \cdot \nabla a = (1, 2y) - \lambda(1,-1) = (1-\lambda; 2y+1) = 0 \iff \lambda = 1; \; y = -1/2$

Impongo $a(x,y) = x - y = 0$: $x = y = -1/2$ e $f(-\frac{1}{2}, -\frac{1}{2}) = -\frac{1}{4}$.

Sia $b(x,y) = x^2+y^2-25$: $\nabla f - \lambda \nabla b = (1, 2y) - \lambda(2x, 2y) =$

$= (1 - 2\lambda x; \; 2y(1-\lambda))$. Risolvo:

$$\begin{cases} 1 - 2\lambda x = 0 \\ 2y(1-\lambda) = 0 \\ x^2+y^2 = 25 \end{cases} \iff \begin{cases} y = 0 \\ x = \pm 5 \\ (\text{solo } x = -5) \end{cases} \text{ o } \begin{cases} \lambda = 1 \; (\text{solo } y = +\sqrt{25 - \frac{1}{4}}) \\ x = 1/2 \\ y = \pm\sqrt{25 - \frac{1}{4}} \end{cases} \Rightarrow f(x,y) = 25 + \frac{1}{4}$$

Devo anche calcolare

$$f\left(\pm\frac{5}{\sqrt{2}}, \pm\frac{5}{\sqrt{2}}\right) = \frac{25}{2} \pm \frac{5}{\sqrt{2}}$$

Confrontando i quattro valori

Trovati, trovo $\max\limits_{\partial M} f = 25+\frac{1}{4}$ e $\min\limits_{\partial M} f = -\frac{1}{4}$

(3) $x'' - 4x' + 3x = e^{3t}\cos(2t)$

Omog. $z'' - 4z' + 3z = 0$. Eq. car.: $0 = \lambda^2 - 4\lambda + 3 = (\lambda-2)^2 - 1$

$\iff \lambda - 2 = \pm 1 \iff \lambda = 2 \pm 1 = 3, 1$

$z(t) = C\cdot e^{3t} + D\cdot e^{t}$: Int. gen. Omog.

Provo con $x(t) = e^{3t}\left[A\cos(2t) + B\sin(2t)\right]$.

$$x'(t) = e^{3t}\cdot\left[3A\cos(2t) + 3B\sin(2t) - 2A\sin(2t) + 2B\cos(2t)\right]$$
$$= e^{3t}\left[(3A+2B)\cos(2t) + (3B-2A)\sin(2t)\right]$$

$$x''(t) = e^{3t}\cdot\left\{\left[3(3A+2B) + 2(3B-2A)\right]\cos(2t) + \left[3(3B-2A) - 2(3A+2B)\right]\sin(2t)\right\}$$
$$= e^{3t}\cdot\left\{(5A+12B)\cos(2t) + (5B-12A)\sin(2t)\right\}$$

$$e^{3t}\cos(2t) = e^{3t}\cdot\left\{\cos(2t)\left[(5A+12B) - 4(3A+2B) + 3A\right] + \sin(2t)\cdot\left[(5B-12A) - 4(3B-2A) + 3B\right]\right\}$$

$$= e^{3t}\cdot\left\{\cos(2t)\cdot(-4A+4B) + \sin(2t)\cdot(-4B-4A)\right\}$$

$$\iff \begin{cases} -4A + 4B = 1 \\ -4B - 4A = 0 \end{cases} \iff \begin{cases} A = -1/8 \\ B = 1/8 \end{cases}$$

$$x(t) = C\cdot e^{3t} + D\cdot e^{t} + \left[\frac{-1}{8}\cos(2t) + \frac{1}{8}\sin(2t)\right]e^{3t}$$

è l'integrale generale; $C, D \in \mathbb{R}$ costanti e $t \in \mathbb{R}$.

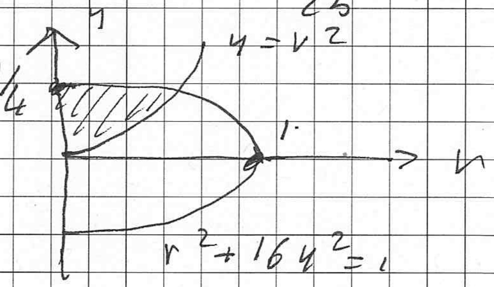
(4) $F(x,y) = x e^{-2y} + h(xy^2, x^2+6, x\sin(y^5))$

Sie $h = h(u,v,w)$

$$\partial_y F(x,y,z) = -2x e^{-2y}$$
$$+ \partial_u h(xy^2, x^2+6, x\cdot\sin(y^5))\cdot 2xy$$
$$+ \partial_w h(xy^2, x^2+6, x\cdot\sin(y^5))\cdot x\cdot\cos(y^5)\cdot 5y^4.$$

(5) $\Omega = \left\{(x,y,z) : \frac{x^2}{25} + 16y^2 + z^2 \le 1;\ y \ge \frac{x^2}{25} + z^2\right\}.$

Con $r^2 = \frac{x^2}{25} + z^2,\ r \ge 0$:



Poi $\begin{cases} y = r^2 \\ r^2 + 16y^2 = 1 \end{cases}$

$16y^2 + y - 1 = 0 \quad y = \frac{-1 \mp \sqrt{65}}{32} \qquad r = \sqrt{\frac{-1+\sqrt{65}}{32}}$

$\begin{cases} x = 5r\cos\theta \\ y = y \\ z = r\sin\theta \end{cases} \quad |\theta| \le \pi$

$dx\,dy\,dz = 5r\,dr\,d\theta$

$$\Rightarrow \iiint_\Omega f(x,y,z)\,dx\,dy\,dz = \int_0^{2\pi} d\theta \int_0^{\sqrt{\frac{-1+\sqrt{65}}{32}}} 5r\,dr \int_{r^2}^{\frac14\sqrt{1-r^2}} dy\, f(5r\cos\theta, y, r\sin\theta)$$

$\boxed{\text{Soluzione per gli Energetici}}$

$a = 0;\ b = \cancel{\frac{-1+\sqrt{65}}{32}}\ \frac14$

$B(y) = \left\{(x,z) : \frac{x^2}{25} + z^2 \le y\right\}$ se $0 \le y \le \frac{-1+\sqrt{65}}{32}$

$B(y) = \left\{(x,z) : \frac{x^2}{25} + z^2 \le 1 - 16y^2\right\}$ se $\frac{-1+\sqrt{65}}{32} \le y \le \frac14$

$\boxed{\text{Soluzione per i Gestionali}}$

(6) $z^3 = 2i + 1 = \sqrt{5} \cdot e^{i \,\mathrm{arctg}\, 2}$

$\iff z = \sqrt[6]{5} \cdot \left[\cos\left(\frac{\mathrm{arctg}\,2 + 2k\pi}{3}\right) + i \sin\left(\frac{\mathrm{arctg}\,2 + 2k\pi}{3}\right)\right]$

con $k = -1, 0, 1$

$z^2 + 3z + 1 = 0 \iff z = \frac{-3 \pm \sqrt{5}}{2}$

(7) (i) Tre soluzioni f, g, h di

$x'' + t^2 \cdot x' + t^3 \cdot x = 0$

sono linearmente dipendenti, ~~poiché~~ cioè $\exists\, a, b, c \in \mathbb{R}$ con $a \neq 0$ o $b \neq 0$ o $c \neq 0$: $\forall t \in \mathbb{R}$: $a f(t) + b g(t) + c h(t) = 0$.

Se so solo che f, g, h sono funzioni, non è detto che ciò sia vero.

P. es. $f(t) = 1$, $g(t) = t$, $h(t) = t^2$.

Se $\forall t \in \mathbb{R}$: $a + bt + ct^2 = 0$, allora l'equazione non è di II grado (quindi $c = 0$), né di primo ($\Rightarrow b = 0$) e dunque $c = 0$: $a = b = c = 0$.

(ii) Se $\exists (x_0, y_0) \in [a, b] \times [c, d]$: $f(x_0, y_0) > 0$, non è detto che $\iint_{[a,b]\times[c,d]} f(x,y)\, dx\, dy > 0$.

P. es. $f(x, y) = x$ soddisfa $f(1, 0) = 1 > 0$, ma

$\int_{-2}^{1} dx \int_{0}^{1} dy\, f(x,y) = \int_{-2}^{1} x\, dx = \frac{1-4}{2} < 0$.

Se ho invece che $\iint_{R = [a,b]\times[c,d]} f(x,y)\, dx\, dy > 0$, da qualche parte $f(x_0, y_0) > 0$: se fosse $f(x_0, y_0) \leq 0$ $\forall (x_0, y_0) \in R$,

$\iint_R f(x,y)\, dx\, dy \leq 0$, contro l'ipotesi.