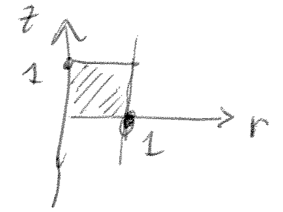
(1) $\Omega = \{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1 ; \quad 0 \leq z \leq 1\}$

$F = (P, Q, R) \in C^1(\Omega, \mathbb{R}^3)$

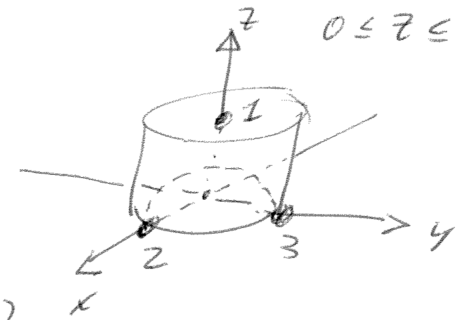
(0) Disegno Ω. Pongo

$$(*) \begin{cases} x = 2r\cos\theta \\ y = 3r\sin\theta \\ z = t \end{cases} \qquad \begin{array}{l} r \in [0, +\infty[\\ \theta \in [0, 2\pi[\\ t \in \mathbb{R} \end{array}$$

Le condizioni che definiscono Ω sono: $r^2 \leq 1$, $0 \leq z \leq 1$



"Ruotando":



(i) $\Sigma_a = \{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} = 1 ; 0 \leq z \leq 1\}$

$\Sigma_b = \{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1 ; \ z = 0\}$

$\Sigma_c = \{x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1 ; \ z = 1\}$

$\Phi_a(\theta, z) = (2\cos\theta, 3\sin\theta, z)$

$\Phi_a : [0, 2\pi[\times [0,1] \to \mathbb{R}^3$

$\Phi_a([0,2\pi[\times [0,1]) = \Sigma_a$

$\Phi_b(x,y) = (x, y, 0)$

$\Phi_c(x,y) = (x, y, 1)$

$\Phi_b : \underset{\mathbb{R}^2}{\underset{\cap}{\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\}}} \to \mathbb{R}^3$

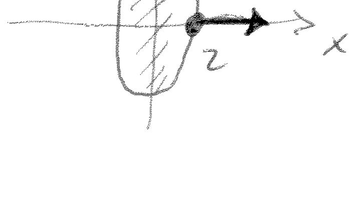
$\Phi_c : \{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\} \to \mathbb{R}^3$

$\Phi_b\left(\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\}\right) = \Sigma_b$

$\Phi_c\left(\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \leq 1\}\right) = \Sigma_c$

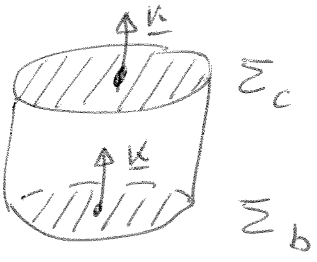
$$(\partial_\theta \Phi_a \times \partial_z \Phi_a)(\theta, z) = \begin{vmatrix} i & j & k \\ -2\sin\theta & 3\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (3\cos\theta, 2\sin\theta, 0)$$

$(\partial_\theta \Phi_a \times \partial_z \Phi_a)(0, z) = (3, 0, 0)$; $\Phi_a(0, z) = (2, 0, z)$



Φ_a è compatibile con l'orientamento di $\Sigma_a \subseteq \partial\Omega$ dato dalla normale esterna ν.

$(\partial_x \Phi_b \times \partial_y \Phi_b)(x,y) = (\partial_x \Phi_c \times \partial_y \Phi_c)(x,y) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \underline{k}$ =



Φ_b non è compatibile con ν

Φ_c è compatibile con ν

(ii) L'integrale che esprime il flusso di F attraverso $\partial\Omega$ è quindi

$$\iint_{\partial\Omega} F \cdot \nu \, d\sigma = \int_0^{2\pi} d\theta \int_0^1 dz \cdot \{ F(2\cos\theta, 3\sin\theta, z) \cdot (3\cos\theta, 2\sin\theta, 0) \}$$

$$- \iint_{\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \le 1\}} F(x,y,0) \cdot (0,0,1) \, dx\, dy + \iint_{\{(x,y) : \frac{x^2}{4} + \frac{y^2}{9} \le 1\}} F(x,y,1) \cdot (0,0,1) \, dx\, dy$$

$$= \int_0^{2\pi} d\theta \int_0^1 dz \cdot \{ P(2\cos\theta, 3\sin\theta, z) \cdot 3\cos\theta + Q(2\cos\theta, 3\sin\theta, z) \cdot 2\sin\theta$$

$$- \iint_{\{(x,y) : \frac{x^2}{4} + y^2/9 \le 1} R(x,y,0) \, dx\, dy + \iint_{\{(x,y) : \frac{x^2}{4} + y^2/9 \le 1} R(x,y,1) \, dx\, dy$$

(iii) $\displaystyle\iint_{\partial\Omega} F_0 \cdot \nu \, d\sigma = \iiint_{\Omega} \operatorname{div} F_0(x,y,z) \cdot dx\, dy\, dz \underset{(\cdot)}{=}$ per il Teo. della divergenza.

con $F_0(x,y,z) = (4xy^2, 9x^2y, z)$, quindi

$\operatorname{div} F_0(x,y,z) = 4y^2 + 9x^2 + 1$,

$$(\cdot) = \iiint_{\Omega} (4y^2 + 9x^2 + 1) \, dx\, dy\, dz = (\cdot\cdot)$$

Uso le coordinate cilindriche (*), e il fatto che

$$\left| \det J \begin{pmatrix} x, y, z \\ r, \theta, t \end{pmatrix} \right| = 2 \cdot 3 \cdot r = 6r,$$

da cui:

$$(60) = \int_0^1 6 r\,dr \int_0^{2\pi} d\theta \int_0^1 dz \left\{ 4\cdot 9 r^2 \sin^2\theta + 9\cdot 4 r^2 \cos^2\theta + 1 \right\}$$

$$= 1\cdot 2\pi \cdot 6 \cdot \int_0^1 (36 r^2 + 1)\, r\, dr = 12\pi \cdot \left(\frac{36}{4} + \frac{1}{2} \right)$$

$$= 114\,\pi .$$

(iv) $$\mathrm{Rot}\,F_0(x,y,z) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 9xy^2 & 4x^2y & z \end{vmatrix}$$

$$= (0,\ 0,\ 8xy - 18xy) = (0,\ 0,\ -10\cdot xy)$$

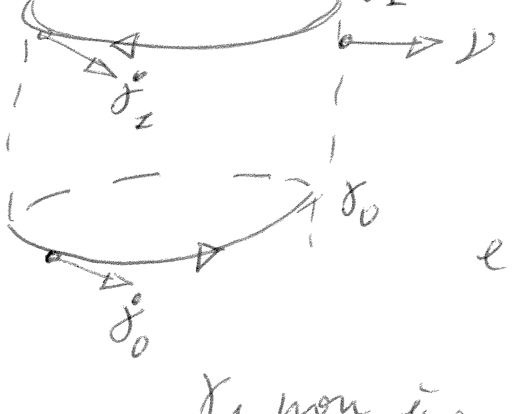
(v) Uso la parametrizzazione Φ_a per $\Sigma = \Sigma_a$, chiaramente compatibile con l'orientazione che ci viene data al punto (v):

$$\iint_\Sigma (\mathrm{Rot}\,F_0)\cdot \nu\, d\sigma = \int_0^{2\pi} d\theta \int_0^1 dz \cdot (0,0,-10\cdot 2\cos\theta\cdot 3\sin\theta)\cdot(3\cos\theta, 2\sin\theta,$$

$$= \int_0^{2\pi} d\theta \int_0^1 dz \cdot 0 = 0 .$$

(vi) $$\partial\Sigma = \left\{(x,y,0) : \frac{x^2}{4} + \frac{y^2}{9} = 1\right\} \cup \left\{(x,y,1) : \frac{x^2}{4} + \frac{y^2}{9} = 0\right\}$$

$$= \Gamma_1 \cup \Gamma_0$$



Siano $\gamma_1 : [0, 2\pi] \to \mathbb{R}^3$

$\gamma_1(\theta) = (2\cos\theta,\ 3\sin\theta,\ 1)$

e $\gamma_0 : [0, 2\pi] \to \mathbb{R}^3$

$\gamma_0(\theta) = (2\cos\theta,\ 3\sin\theta,\ 0)$

$$\left.\begin{array}{l} \gamma_1 \text{ non è compatibile} \\ \gamma_0 \text{ è compatibile} \end{array}\right\} \text{con l'orientamento } \nu.$$

(2) Risolvere

(E) $y'' + 16 \cdot y = e^{4x} + e^{-4x} + 2x$.

Considero l'eq. omogenea associata

(O) $z'' + 16 \cdot z = 0$

e la sua eq. caratteristica:

$\lambda^2 + 16 = 0 \qquad (\iff \lambda = \pm 4i)$.

$IG(O) = \{ z(t) = c_1 \cos(4x) + c_2 \cdot \sin(4x),\ c_1, c_2 \in \mathbb{R} \}$

$\cap\!\!\!\!\;\!\;\shortmid$ $C^1(\mathbb{R}, \mathbb{R})$.

Per (E), provo con $y(x) = A e^{4x} + B e^{-4x} + Cx + D$

$y'(x) = 4A e^{4x} - 4B e^{-4x} + C$

$y''(x) = 16A \cdot e^{4x} + 16 B e^{-4x}$, quindi

(E) vale per $y \iff e^{4x} + e^{-4x} + 2x =$

$= (16 A e^{4x} + 16 B e^{-4x}) + 16 (A e^{4x} + B e^{-4x} + Cx + D)$

$= 32 A e^{4x} + 32 B e^{-4x} + 16 C x + 16 D$

$\iff A = B = \frac{1}{32},\ D = 0,\ C = 1/16$.

$IG(E) = \{ y(t) = c_1 \cdot \cos(4x) + c_2 \cdot \sin(4x) + \frac{e^{4x} + e^{-4x}}{32} + \frac{x}{16} \}$

$\cap\!\!\!\!\;\!\;\shortmid$ $C^2(\mathbb{R})$