# Test di prove

① $f(x) = |x| \cdot e^{-|x-1|}$

Dominio$(f) = \mathbb{R}$, $f$ è continua su $\mathbb{R}$

e $\exists f'(x)$ $\forall x \in \mathbb{R} \setminus \{0, 1\}$.

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{|x|}{e^{|x-1|}} = 0$$

per il confronto polinomi/esponenziali e per il fatto che $|x-1| \underset{x \to \pm\infty}{\sim} |x|$.

Calcolo $\forall x \neq 0,1$: $f'(x) = sgn(x) \cdot e^{-|x-1|} + |x| \cdot e^{-|x-1|} \cdot [-sgn(x-1)]$

$= e^{-|x-1|} \cdot [sgn(x) - |x| \cdot sgn(x-1)] =$

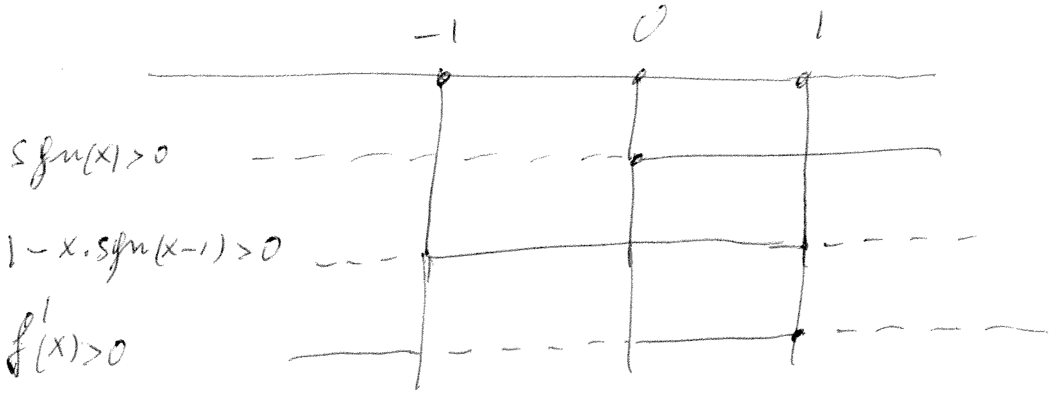
$= e^{-|x-1|} \cdot sgn(x) \cdot [1 - x \cdot sgn(x-1)]$ poiché $|x| = sgn(x) \cdot x$.

- $e^{-|x-1|} > 0$ $\forall x$
- $sgn(x) > 0 \iff x > 0$
- $1 - x \cdot sgn(x-1) = \left\{\begin{array}{ll} 1-x & se \ x-1>0 \\ 1+x & se \ x-1<0 \end{array}\right\} > 0 \iff$

$\begin{cases} 1-x>0 \\ x-1>0 \end{cases}$ oppure $\begin{cases} 1+x>0 \\ x-1<0 \end{cases} \iff \begin{cases} 1>x \\ x>1 \end{cases}$ o $\begin{cases} x>-1 \\ x<1 \end{cases}$

impossibile

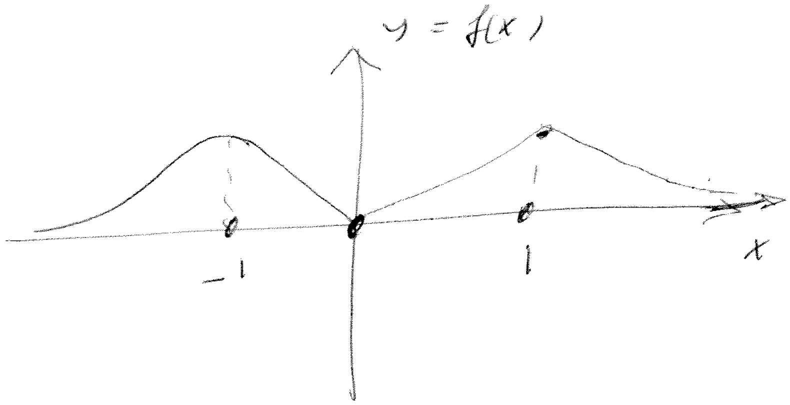
$\iff -1 < x < 1$.



$f'(x) > 0 \iff x < -1$ o $0 < x < 1$,

$f$ è crescente (usando la sua continuità) in $]-\infty, -1]$ e in $[0, 1]$.

<u>Grafico</u>:




<u>$f$ è derivabile in $x = 0$?</u> $f'(0^+) = e^{-1} \neq -e^{-1} = f'(0^-)$: <u>no</u>

<u>$f$ è derivabile in $x = 1$?</u> $f'(1^+) = 0 \neq 2 = f'(1^-)$: <u>no</u>

(2) Pongo $y = \sin(2x)$, $dy = \frac{d\sin(2x)}{dx}\,dx = 2\cdot\cos(2x)\,dx$,

$x = 0 \iff y = \sin(2\cdot 0) = 0$

$x = \pi/8 \iff y = \sin\left(2\cdot\frac{\pi}{8}\right) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$ :

$$I = \int_0^{\pi/8} \frac{\sin(2x)\cdot\cos(2x)}{\sin^2(2x) + 2\cdot\sin(2x) + 10}\,dx = \int_0^{1/\sqrt{2}} \frac{y}{y^2+2y+10}\,\frac{dy}{2}.$$

Se $P(y) = y^2 + 2y + 10$, $\Delta = (-2)^2 - 4\cdot 10 = -36 < 0$.

Scrivo $P(y) = y^2 + 2y + 1 + 9 = (y+1)^2 + 9 = 9\cdot\left(\frac{(y+1)^2}{9} + 1\right)$
$= 9\cdot\left[\left(\frac{y+1}{3}\right)^2 + 1\right]$ : pongo $z = \frac{y+1}{3}$, $dz = \frac{dy}{3}$,

$y\Big|_0^{1/\sqrt{2}} \iff z\Big|_{1/3}^{(1+1/\sqrt{2})/3}$.

oss. che $y = 3z - 1$

Quindi,

$$I = \frac{1}{2}\cdot \int_{1/3}^{(1+1/\sqrt{2})/3} \frac{3z-1}{9\cdot(z^2+1)}\, 3\,dz = \frac{1}{6}\int_{1/3}^{(1+1/\sqrt{2})/3} \frac{3z-1}{z^2+1}\,dz$$

$$= \frac{1}{2}\int_{1/3}^{(1+1/\sqrt{2})/3} \frac{z}{z^2+1}\,dz - \frac{1}{6}\int_{1/3}^{(1+1/\sqrt{2})/3} \frac{dz}{z^2+1}.$$

$$= \boxed{\frac{1}{4}\left[\log(z^2+1)\right]_{1/3}^{(1+1/\sqrt{2})/3} - \frac{1}{6}\left[\operatorname{arctg}(z)\right]_{1/3}^{(1+1/\sqrt{2})/3}}$$

(3) Se $x \to 0$, anche $\sin(2x) \to 0$, quindi, al fine di sviluppare il numeratore all'ordine 3,

$$e^{\sin(2x)} - e^{2x} = 1 + \sin(2x) + \frac{\sin^2(2x)}{2} + \frac{\sin^3(3x)}{6} + o(\sin^3(2x))$$
$$-\left(1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + o(x^3)\right)$$

$$= 1 + \left(2x - \frac{(2x)^3}{6} + o(x^3)\right) + \frac{1}{2}\cdot\left(2x + o(x^2)\right)^2 + \frac{1}{6}\left(2x + o(x^2)\right)^3$$
$$+ o(\sin^3(2x)) - 1 - 2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{6} + o(x^3).$$

$$= 1 + 2x - \frac{(2x)^3}{6} + \frac{1}{2}\cdot(2x)^2 + \frac{1}{6}(2x)^3 - 1 - 2x - \frac{(2x)^2}{2} - \frac{(2x)^3}{6}$$
$$+ o(\sin^3(2x)) + o(x^3)$$

$$= -\frac{8x^3}{6} + o(x^3) + o(\sin^3(2x)) = -\frac{8}{6}x^3 + o(x^3)$$
$$= \left(-\frac{4}{3} + o(1)\right)\cdot x^3$$

essendo $\sin(2x) \underset{x\to 0}{\sim} 2x$

Dunque

$$\frac{e^{\sin(2x)} - e^{2x}}{x^3} = \frac{\left(-4/3 + o(1)\right) x^3}{x^3} = -\frac{4}{3} + o(1) \underset{x \to 0+}{\longrightarrow} -\frac{4}{3}$$

che è il limite cercato.