Abstract

For $0 \leq \sigma < 1/2$ we characterize Carleson measures μ for the analytic Besov-Sobolev spaces B_2^σ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ by the discrete tree condition

$$\sum_{\beta \geq \alpha} \left[2^{\sigma d(\beta)} I^* \mu(\beta) \right]^2 \leq C I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n,$$

on the associated Bergman tree $\mathcal{T}_n$. Combined with recent results about interpolating sequences this leads, for this range of σ, to a characterization of universal interpolating sequences for B_2^σ and also for its multiplier algebra.

However, the tree condition is *not* necessary for a measure to be a Carleson measure for the Drury-Arveson Hardy space $H_n^2 = B_2^{1/2}$. We show that μ is a Carleson measure for $B_2^{1/2}$ if and only if both the simple condition

$$2^{d(\alpha)} I^* \mu(\alpha) \leq C, \qquad \alpha \in \mathcal{T}_n,$$

and the split tree condition

$$\sum_{k \geq 0} \sum_{\gamma \geq \alpha} 2^{d(\gamma)-k} \sum_{(\delta, \delta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\delta) \, I^* \mu(\delta') \leq C I^* \mu(\alpha), \qquad \alpha \in \mathcal{T}_n,$$

hold. This gives a sharp estimate for Drury's generalization of von Neumann's operator inequality to the complex ball, and also provides a universal characterization of Carleson measures, up to dimensional constants, for Hilbert spaces with a complete continuous Nevanlinna-Pick kernel function.

We give a detailed analysis of the split tree condition for measures supported on embedded two manifolds and we find that in some generic cases the condition simplifies. We also establish a connection between function spaces on embedded two manifolds and Hardy spaces of plane domains.

Carleson Measures for the Drury-Arveson Hardy space and other Besov-Sobolev Spaces on Complex Balls

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1 Overview

We give a description of Carleson measures for certain Hilbert spaces of holomorphic functions on the ball in $\mathbb{C}^n$. In the next section we give background and a summary. We also describe ways the characterization can be used and how the characterization simplifies in some special cases. In the following section we collect certain technical tools. The main work of characterizing the Carleson measures is in the section after that. A brief final appendix has the real variable analog of our main results.

2 Introduction

2.1 Function Spaces

Let $\mathbb{B}_n$ be the unit ball in $\mathbb{C}^n$. Let dz be Lebesgue measure on $\mathbb{C}^n$ and let $d\lambda_n(z) = (1-|z|^2)^{-n-1}dz$ be the invariant measure on the ball. For integer $m \geq 0$, and for $0 \leq \sigma < \infty$, $1 < p < \infty$, $m + \sigma > n/p$ we define the analytic Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ to consist of those holomorphic functions f on the ball such that

$$\left\{ \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^p + \int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right\}^{\frac{1}{p}} < \infty. \qquad (1)$$

Here $f^{(m)}$ is the m^{th} order complex derivative of f. The spaces $B_p^\sigma(\mathbb{B}_n)$ are independent of m and are Banach spaces with norms given in (1).

For $p = 2$ these are Hilbert spaces with the obvious inner product. This scale of spaces includes the Dirichlet spaces $B_2(\mathbb{B}_n) = B_2^0(\mathbb{B}_n)$, weighted Dirichlet-type spaces with $0 < \sigma < 1/2$, the Drury-Arveson Hardy spaces $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$ (also known as the symmetric Fock spaces over $\mathbb{C}^n$) ([22], [12], [20]), the Hardy spaces $H^2(\mathbb{B}_n) = B_2^{n/2}(\mathbb{B}_n)$, and the weighted Bergman spaces with $\sigma > n/2$.

Alternatively these Hilbert spaces can be viewed as part of the Hardy-Sobolev scale of spaces $J_\gamma^2(\mathbb{B}_n)$, $\gamma \in \mathbb{R}$, consisting of all holomorphic functions f in the unit ball whose radial derivative $R^\gamma f$ of order γ belongs to the Hardy space $H^2(\mathbb{B}_n)$ ($R^\gamma f = \sum_{k=0}^\infty (k+1)^\gamma f_k$ if $f = \sum_{k=0}^\infty f_k$ is the homogeneous expansion of f). The Hardy-Sobolev scale coincides with the Besov-Sobolev scale and we have

$$B_2^\sigma(\mathbb{B}_n) = J_\gamma^2(\mathbb{B}_n), \quad \sigma + \gamma = \frac{n}{2}, 0 \le \sigma \le \frac{n}{2}.$$

Thus σ measures the order of antiderivative required to belong to the Dirichlet space $B_2(\mathbb{B}_n)$, and $\gamma = \frac{n}{2} - \sigma$ measures the order of the derivative that belongs to the Hardy space $H^2(\mathbb{B}_n)$.

2.2 Carleson Measures

By a Carleson measure for $B_p^\sigma(\mathbb{B}_n)$ we mean a positive measure defined on $\mathbb{B}_n$ such that the following Carleson embedding holds; for $f \in B_p^\sigma(\mathbb{B}_n)$

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu \le C_\mu \|f\|^p_{B_p^\sigma(\mathbb{B}_n)}. \tag{2}$$

The set of all such is denoted $CM(B_p^\sigma(\mathbb{B}_n))$ and we define the Carleson measure norm $\|\mu\|_{Carleson}$ to be the infimum of the possible values of $C_\mu^{1/p}$. In [10] we described the Carleson measures for $B_p^\sigma(\mathbb{B}_n)$ for $\sigma = 0$ and $1 < p < 2 + \frac{1}{n-1}$. Here we consider $\sigma > 0$ and focus our attention on the Hilbert space cases, $p = 2$. We show that, *mutatis mutandis,* the results for $\sigma = 0$ extend to the range $0 \le \sigma < 1/2$. Fundamental to this extension is the fact that for $0 \le \sigma < 1/2$ the real part of the reproducing kernel for $B_2^\sigma(\mathbb{B}_n)$ is comparable to its modulus. In fact, in the appendix when we use the modulus of the reproducing kernel in defining nonisotropic potential spaces, results similar to those for $\sigma = 0$ continue to hold for $0 \le \sigma < n/2$. However even though the reproducing kernel for $B_2^{1/2}(\mathbb{B}_n) = H_n^2$ has positive real part, its real part is not comparable to its modulus. For that space a new type of analysis must be added and by doing that we describe the Carleson measures for $B_2^{1/2}(\mathbb{B}_n)$. For $1/2 < \sigma < n/2$ the real part of the kernel is not positive, our methods don't apply, and that range remains mysterious. For $\sigma \ge n/2$ we are in the realm of the classical Hardy and Bergman spaces and the description of the Carleson measures is well established [31], [36].

Let $\mathcal{T}_n$ denote the Bergman tree associated to the ball $\mathbb{B}_n$ as in [10]. We show (Theorem 23) that the tree condition,

$$\sum_{\beta \ge \alpha} \left[2^{\sigma d(\beta)} I^* \mu(\beta) \right]^2 \le C I^* \mu(\alpha) < \infty, \quad \alpha \in \mathcal{T}_n, \tag{3}$$

characterizes Carleson measures for the Besov-Sobolev space $B_2^\sigma(\mathbb{B}_n)$ in the range $0 \le \sigma < 1/2$. To help place this condition in context we compare it with

the corresponding simple condition. The condition

$$2^{2\sigma d(\alpha)} I^* \mu(\alpha) \leq C \qquad (\mathrm{SC}(\sigma))$$

is necessary for μ to be a Carleson measure as is seen by testing the embedding (2) on reproducing kernels or by starting with (3) and using the infinite sum there to dominate the single term with $\beta = \alpha$. Although $\mathrm{SC}(\sigma)$ is not sufficient to insure that μ is a Carleson measure, slight strengthenings of it are sufficient, see Lemma 32 below. In particular, for any $\varepsilon > 0$ the condition $\mathrm{SC}(\sigma + \varepsilon)$ is sufficient.

On the other hand if $\sigma \geq 1/2$ then, by the results in [18] together with results in the Appendix, there are positive measures μ on the ball that are Carleson for $J^2_{\frac{n}{2}-\sigma}(\mathbb{B}_n) = B^\sigma_2(\mathbb{B}_n)$ but fail to satisfy the tree condition (3). Our analysis of Carleson measures for the "endpoint" case $B^{1/2}_2(\mathbb{B}_n)$, the Drury-Arveson Hardy space, proceeds in two stages. First, by a functional analytic argument we show that the norm $\|\mu\|_{Carleson}$ is comparable, independently of dimension, with the best constant C in the inequality

$$\int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \left(\mathrm{Re}\, \frac{1}{1 - \overline{z} \cdot z'} \right) f(z')\, d\mu(z')\, g(z)\, d\mu(z) \leq C \|f\|_{L^2(\mu)} \|g\|_{L^2(\mu)}. \qquad (4)$$

We then proceed with a geometric analysis of the conditions under which this inequality holds. If in (4) we were working with the integration kernel $\left| \frac{1}{1-\overline{z}\cdot z'} \right|$ rather than $\mathrm{Re}\, \frac{1}{1-\overline{z}\cdot z'}$ we could do an analysis similar to that for $\sigma < 1/2$ and would find the inequality characterized by the tree condition with $\sigma = 1/2$:

$$\sum_{\beta \geq \alpha} 2^{d(\beta)} I^* \mu(\beta)^2 \leq C I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n, \qquad (5)$$

see Subsection 4.3 and the Appendix. However, as we will show, for $n > 1$, the finiteness of $\|\mu\|_{Carleson}$ is equivalent *neither* to the tree condition (5), *nor* to the simple condition

$$2^{d(\alpha)} I^* \mu(\alpha) \leq C, \qquad \alpha \in \mathcal{T}_n, \qquad (6)$$

($\mathrm{SC}(1/2)$ in the earlier notation).

To proceed we will introduce additional structure on the Bergman tree $\mathcal{T}_n$. For α in $\mathcal{T}_n$, we denote by $[\alpha]$ an equivalence class in a certain quotient tree $\mathcal{R}_n$ of "rings" consisting of elements in a "common slice" of the ball having common distance from the root. Using this additional structure we will show in Theorem 34 that the Carleson measures are characterized by the simple condition (6) together with the "split" tree condition

$$\sum_{k \geq 0} \sum_{\gamma \geq \alpha} 2^{d(\gamma)-k} \sum_{(\delta,\delta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\delta)\, I^* \mu(\delta') \leq C I^* \mu(\alpha), \qquad \alpha \in \mathcal{T}_n, \qquad (7)$$

and moreover we have the norm estimate

$$\|\mu\|_{Carleson} \approx \sup_{\alpha\in\mathcal{T}_n} \sqrt{2^{d(\alpha)} I^*\mu\,(\alpha)} \tag{8}$$
$$+ \sup_{\substack{\alpha\in\mathcal{T}_n \\ I^*\mu(\alpha)>0}} \sqrt{\frac{1}{I^*\mu\,(\alpha)} \sum_{k\geq 0}\sum_{\gamma\geq\alpha} 2^{d(\gamma)-k} \sum_{(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)} I^*\mu\,(\delta)\, I^*\mu\,(\delta')}.$$

The restriction $(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)$ in the sums above means that we sum over all pairs (δ,δ') of grand^k-children of γ that have γ as their minimum, that lie in well-separated rings in the quotient tree, but whose predecessors of order two, $A^2\delta$ and $A^2\delta'$, lie in a common ring. That is, the ring tree geodesics to δ and to δ' have recently split, at distance roughly k from γ. Note that if we were to extend the summation to all pairs (δ,δ') of grand^k-children of γ then this condition would be equivalent to the tree condition (5). More formally,

Definition 1 *The set $\mathcal{G}^{(k)}(\gamma)$ consists of pairs (δ,δ') of grand^k-children of γ in $\mathcal{G}^{(k)}(\gamma)\times\mathcal{G}^{(k)}(\gamma)$ which satisfy $\delta\wedge\delta'=\gamma$, $\left[A^2\delta\right]=\left[A^2\delta'\right]$ (which implies $d\left([\delta],[\delta']\right)\leq 4$) and $d^*\left([\delta],[\delta']\right)=4$.*

Here

$$d^*\left([\alpha],[\beta]\right)=\inf_{U\in\mathcal{U}_n} d\left(\left[t\left(Uc_\alpha\right)\right],\left[t\left(Uc_\beta\right)\right]\right),$$

and $\mathcal{U}_n$ denotes the group of unitary rotations of the ball $\mathbb{B}_n$. For α in the Bergman tree $\mathcal{T}_n$, c_α is the "center" of the Bergman kube K_α. For $z\in\mathbb{B}_n$, $t(z)$ denotes the element $\alpha'\in\mathcal{T}_n$ such that $z\in K_{\alpha'}$. Thus $d^*\left([\alpha],[\beta]\right)$ measures an "invariant" distance between the rings $[\alpha]$ and $[\beta]$. Note that $\mathcal{G}^{(0)}(\gamma)=\mathcal{G}(\gamma)$ is the set of grandchildren of γ. Further details can be found in Subsection 4.2.1 below on a modified Bergman tree and its quotient tree.

We noted before that for $0\leq\sigma<1/2$ the tree condition (3) implies the corresponding simple condition SC(σ). However the split tree condition (7) does not imply the simple condition (6). In fact, measures supported on a slice, i.e., on the intersection of the ball with a complex line through the origin, satisfy the split tree condition vacuously. This is because for measure supported on a single slice and δ and δ' in different rings at most one of the factors $I^*\mu\,(\delta)$, $I^*\mu\,(\delta')$ can be nonzero. However such measures may or may not satisfy (6). Similarly the split tree condition is vacuously satisfied when $n=1$. In that case we have the classical Hardy space and Carleson's classical condition SC(1/2).

In our proof of (8) the implicit constants of equivalence depend on the dimension n. One reason for attention to possible dimensional dependence of constants arises in Subsubsection 2.3.4. Roughly, a large class of Hilbert spaces with reproducing kernels have natural realizations as subspaces of the various H^2_n and this occurs in ways that lets us use the characterization of Carleson measures for H^2_n to obtain descriptions of the Carleson measures for these other spaces. However in the generic case, as well as for the most common examples, $n=\infty$. When $n=\infty$ we can pull back characterizations of Carleson measures of the

form (2) or (4) but, because of the dimensional dependence of the constants, we cannot obtain characterizations using versions of (6) and (7).

Finally, we mention two technical refinements of these results. First, it suffices to test the bilinear inequality (4) over $f = g = \chi_{T(w)}$ where $T(w)$ is a nonisotropic Carleson region with vertex w. This holds because in Subsection 4.2.4, when proving the necessity of the split tree condition, we only use that special case of the bilinear inequality. However that observation commits us to a chain of implications which uses (8) and thus we don't know that the constants in the restricted condition are independent of dimension. Second, the condition (7) can be somewhat simplified by further restricting the sum over k and γ on the left side to $k \leq \varepsilon d(\gamma)$ for any fixed $\varepsilon > 0$; the resulting ε-split tree condition is

$$\sum_{\gamma \geq \alpha : 0 \leq k \leq \epsilon d(\gamma)} 2^{d(\gamma)-k} \sum_{(\delta,\delta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\delta)\, I^* \mu(\delta') \leq C I^* \mu(\alpha), \qquad \alpha \in \mathcal{T}_n. \quad (9)$$

The reason (6) and (9) suffice is that the sum in (7) over $k > \varepsilon d(\gamma)$ is dominated by the left side of (3) with $\sigma = (1-\varepsilon)/2$, and that this condition is in turn implied by the simple condition (6). See Lemma 32 below.

Finally, as we mentioned, the characterization of Carleson measures for $B_2^\sigma(\mathbb{B}_n)$ remains open in the range $1/2 < \sigma < n/2$. The Carleson measures for the Hardy space, $\sigma = n/2$, and the weighted Bergman spaces, $\sigma > n/2$, are characterized by SC(σ); see [31] and [36].

2.3 Applications and special cases

Before proving the characterizations of Carleson measures we present some uses of those results and also describe how the general results simplify in some cases. In doing this we will use the results and notation of later sections but we will not use results from this section later.

We describe the multiplier algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ of $B_2^\sigma(\mathbb{B}_n)$ for $0 \leq \sigma \leq 1/2$. For the smaller range $0 \leq \sigma < 1/2$ we describe the interpolating sequences for $B_2^\sigma(\mathbb{B}_n)$ and for $M_{B_2^\sigma(\mathbb{B}_n)}$. We give an explicit formula for the norm which arises in Drury's generalization of von Neumann's operator inequality to the complex ball $\mathbb{B}_n$. We give a universal characterization of Carleson measures for Hilbert spaces with a complete Nevanlinna-Pick kernel function.

To understand the split tree condition (7) better we investigate the structure of the Carleson measures for $B_2^{1/2}(\mathbb{B}_n)$ which are supported on real 2-manifolds embedded in $\mathbb{B}_n$. This will also give information about Carleson measures for spaces of functions on those manifolds. Suppose we have a C^1 embedding of a real 2-manifold $\mathcal{S}$ into $\mathbb{B}_n$ and that $\bar{\mathcal{S}}$ meets the boundary of the ball transversally in a curve Γ. Suppose we have a Carleson measure for $B_2^{1/2}(\mathbb{B}_n)$ supported in $\mathcal{S}$. We find that

- If Γ is transverse to the complex tangential boundary directions then (9) becomes vacuous for small ε and the Carleson measures are described by

the simple condition (6). In particular this applies to C^1 embedded holomorphic curves and shows that the Carleson measures for the associated spaces coincide with the Carleson measures for the Hardy spaces of those curves. For planar domains we show that if the embedding is C^2 then these spaces coincide with the Hardy spaces.

- If Γ is a complex tangential curve, that is if its tangent lies in the complex tangential boundary directions then (9) reduces to the tree condition (5) and the Carleson measures are described by the tree condition. A similar result holds for measures supported on embedded real k-manifolds which meet the boundary transversely and in the complex tangential directions.

On the other hand, the embedding $\mathcal{S}$ of the unit disk into $\mathbb{B}_\infty$ associated with a space $B_2^\sigma(\mathbb{B}_1)$, $0 \leq \sigma < 1/2$, extends to $\bar{\mathcal{S}}$, is Lipschitz continuous of order σ but not C^1 and is not transverse to the boundary. In this more complicated situation neither of the two simplifications occur.

2.3.1 Multipliers

A holomorphic function f on the ball is called a multiplier for the space $B_2^\sigma(\mathbb{B}_n)$ if the multiplication operator M_f defined by $M_f(g) = fg$ is a bounded linear operator on $B_2^\sigma(\mathbb{B}_n)$. In that case the multiplier norm of f is defined to be the operator norm of M_f. The space of all such is denoted $M_{B_2^\sigma(\mathbb{B}_n)}$.

Ortega and Fabrega [30] have characterized multipliers for the Hardy-Sobolev spaces using Carleson measures. We refine their result by including a geometric characterization of those measures.

Theorem 2 *Suppose $0 \leq \sigma \leq 1/2$. Then f is in $M_{B_2^\sigma(\mathbb{B}_n)}$ if and only if f is bounded and for some, equivalently for any, $k > n/2 - \sigma$*

$$d\mu_{f,k} = \left|(1-|z|^2)^k f^{(k)}\right|^2 (1-|z|^2)^{2\sigma} d\lambda_n(z) \in CM(B_2^\sigma(\mathbb{B}_n)).$$

In that case we have

$$\|f\|_{M_{B_2^\sigma(\mathbb{B}_n)}} \sim \|f\|_{H^\infty(\mathbb{B}_n)} + \|d\mu_{f,k}\|_{CM(B_2^\sigma(\mathbb{B}_n))}.$$

If $0 \leq \sigma < 1/2$ the second summand can be evaluated using Theorem 23. For $\sigma = 1/2$ the second summand can be evaluated using Theorem 34.

In the familiar case of the one variable Hardy space, $n = 1$, $\sigma = 1/2$, and $k = 1$; the Carleson measure condition need not be mentioned because it is implied by the boundedness of f, for instance because of the inclusion $H^\infty(\mathbb{B}_1) \subset BMO(\mathbb{B}_1)$ and the characterization of BMO in terms of Carleson measures. Thus the multiplier algebra consists of all bounded functions. However for $n > 1$ and $0 \leq \sigma \leq 1/2$ as well as $n = 1$ and $0 \leq \sigma < 1/2$, there are bounded functions which are not multipliers. Because the constant functions are in all the B_2^σ we can establish this by exhibiting bounded functions not in

the B_2^σ. In [20] Chen constructs such functions for $n > 1$, $\sigma = 1/2$. If $\sigma < 1/2$ then $B_2^\sigma \subset B_2^{1/2}$ and hence Chen's functions also fail to be in B_2^σ. Similar but simpler examples work for $n = 1, 0 \leq \sigma < 1/2$. Other approaches to this are in [22] and [12].

2.3.2 Interpolating sequences

Given σ, $0 \leq \sigma < 1/2$ and a discrete set $Z = \{z_i\}_{i=1}^\infty \subset \mathbb{B}_n$ we define the associated measure $\mu_Z = \sum_{j=1}^\infty (1-|z_j|^2)^{2\sigma} \delta_{z_j}$. We say that Z is an interpolating sequence for $B_2^\sigma(\mathbb{B}_n)$ if the restriction map R defined by $Rf(z_i) = f(z_i)$ for $z_i \in Z$ maps $B_2^\sigma(\mathbb{B}_n)$ into and onto $\ell^2(Z, \mu_Z)$. We say that Z is an interpolating sequence for $M_{B_2^\sigma(\mathbb{B}_n)}$ if R maps $M_{B_2^\sigma(\mathbb{B}_n)}$ into and onto $\ell^\infty(Z, \mu_Z)$. Using results of B. Böe [17], J. Agler and J. E. M^{c}Carthy [2], D. Marshall and C. Sundberg [27], along with the above Carleson measure characterization for $B_2^\sigma(\mathbb{B}_n)$ we now characterize those sequences. Denote the Bergman metric on the complex ball $\mathbb{B}_n$ by β.

Theorem 3 *Suppose σ, Z, and μ_Z are as described. Then Z is an interpolating sequence for $B_2^\sigma(\mathbb{B}_n)$* if and only if *Z is an interpolating sequence for the multiplier algebra $M_{B_2^\sigma(\mathbb{B}_n)}$* if and only if *$Z$ satisfies the separation condition $\inf_{i \neq j} \beta(z_i, z_j) > 0$ and μ_Z is a $B_2^\sigma(\mathbb{B}_n)$ Carleson measure, i.e. it satisfies the tree condition (3).*

Proof. The case $\sigma = 0$ was proved in [27] when $n = 1$ and in [10] when $n > 1$. If $0 < \sigma < 1/2$, then Corollary 1.12 of [2] shows that the reproducing kernel $k(z, w) = \left(\frac{1}{1-\overline{w}\cdot z}\right)^{2\sigma}$ has the complete Nevanlinna-Pick property. Indeed, the corollary states that k has the complete Nevanlinna-Pick property if and only if for any finite set $\{z_1, z_2, ..., z_m\}$, the matrix H_m of reciprocals of inner products of reproducing kernels k_{z_i} for z_i, i.e.

$$H_m = \left[\frac{1}{\langle k_{z_i}, k_{z_j}\rangle}\right]_{i,j=1}^m,$$

has exactly one positive eigenvalue counting multiplicities. We may expand $\langle k_{z_i}, k_{z_j}\rangle^{-1}$ by the binomial theorem as

$$(1 - \overline{z_j}\cdot z_i)^{2\sigma} = 1 - \sum_{\ell=1}^\infty c_\ell (\overline{z_j}\cdot z_i)^\ell,$$

where $c_\ell = (-1)^{\ell+1} \begin{pmatrix} 2\sigma \\ \ell \end{pmatrix} \geq 0$ for $\ell \geq 1$ and $0 < 2\sigma < 1$. Now the matrix $[\overline{z_j}\cdot z_i]_{i,j=1}^m$ is nonnegative semidefinite since

$$\sum_{i,j=1}^m \zeta_i (\overline{z_j}\cdot z_i)\overline{\zeta_i} = |(\zeta_1 z_1, ..., \zeta_m z_m)|^2 \geq 0.$$

Thus by Schur's theorem so is $\left[\left(\overline{z_j}\cdot z_i\right)^\ell\right]_{i,j=1}^m$ for every $\ell \geq 1$, and hence, also, so is the sum with positive coefficients. Thus the positive part of the matrix H_m is $[1]_{i,j=1}^m$ which has rank 1, and hence the sole positive eigenvalue of H_m is m. Once we know that $B_2^\sigma(\mathbb{B}_n)$ has the Pick property then it follows from a result of Marshall and Sundberg (Theorem 9.19 of [3]) that the interpolating sequences for $M_{B_2^\sigma(\mathbb{B}_n)}$ are the same as those for $B_2^\sigma(\mathbb{B}_n)$. Thus we need only consider the case of $B_2^\sigma(\mathbb{B}_n)$.

We now invoke a theorem of B. Böe [17] which says that for certain Hilbert spaces with reproducing kernel, in the presence of the separation condition (which is necessary for an interpolating sequence, see Ch. 9 of [3]) a necessary and sufficient condition for a sequence to be interpolating is that the Grammian matrix associated with Z is bounded. That matrix is built from normalized reproducing kernels; it is

$$\left[\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|}\right\rangle\right]_{i,j=1}^\infty . \tag{10}$$

The spaces to which Böe's theorem applies are those where the kernel has the complete Nevanlinna-Pick property, which we have already noted holds in our case, and which have the following additional technical property. Whenever we have a sequence for which the matrix (10) is bounded on ℓ^2 then the matrix with absolute values

$$\left[\left|\left\langle \frac{k_{z_i}}{\|k_{z_i}\|}, \frac{k_{z_j}}{\|k_{z_j}\|}\right\rangle\right|\right]_{i,j=1}^\infty$$

is also bounded on ℓ^2. This property holds in our case because, for σ in the range of interest, $\mathrm{Re}\left(\frac{1}{1-\overline{z_j}\cdot z_i}\right)^{2\sigma} \approx \left|\frac{1}{1-\overline{z_j}\cdot z_i}\right|^{2\sigma}$ which, as noted in [17], insures that the Gramm matrix has the desired property. (It is this step that precludes considering $\sigma = 1/2$.) Finally, by Proposition 9.5 of [3], the boundedness on ℓ^2 of the Grammian matrix is equivalent to $\mu_Z = \sum_{j=1}^\infty \left\|k_{z_j}\right\|^{-2}\delta_{z_j} = \sum_{j=1}^\infty (1-|z_j|^2)^{2\sigma}\delta_{z_j}$ being a Carleson measure. Thus the obvious generalization to higher dimensions of the interpolation theorem of Böe in [17] completes the proof. (Böe presents his work in dimension $n = 1$, but, as he notes, the proof extends to spaces with the above properties.)

(When we defined "interpolating sequence" we required that R map into and onto $\ell^2(Z,\mu_Z)$. In the most well known case, the classical Hardy space, $n = 1, \sigma = 1/2$, if R is onto it must be into. However for the classical Dirichlet space the map can be onto without being into. Hence one can ask for a characterization of those maps for which R is onto. The question is open; partial results are in [15], [17], and [11].)

2.3.3 The Drury-Arveson Hardy space and von Neumann's inequality

It is a celebrated result of von Neumann [28] that if T is a contraction on a Hilbert space and f is a complex polynomial then $\|f(T)\| \leq \sup\{|f(\gamma)| : |\gamma| = 1\}$. An extension of this to n-tuples of operators was given by Drury [22]. Let $A = (A_1, ..., A_n)$ be an n-(row)-contraction on a complex Hilbert space $\mathcal{H}$, i.e. an n-tuple of commuting linear operators on $\mathcal{H}$ satisfying

$$\sum_{j=1}^{n} \|A_j h\|^2 \leq \|h\|^2 \text{ for all } h \in \mathcal{H}.$$

Drury showed in [22] that if f is a complex polynomial on $\mathbb{C}^n$ then

$$\sup_{\substack{A \text{ an} \\ n\text{-contraction}}} \|f(A)\| = \|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}, \tag{11}$$

where $\|f(A)\|$ is the operator norm of $f(A)$ on $\mathcal{H}$, and $\|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}$ denotes the multiplier norm of the polynomial f on Drury's Hardy space of holomorphic functions

$$\mathcal{K}(\mathbb{B}_n) = \left\{ \sum_k a_k z^k,\ z \in \mathbb{B}_n : \sum_k |a_k|^2 \frac{k!}{|k|!} < \infty \right\}.$$

This space is denoted H_n^2 by Arveson in [12] (who also proves (11) in Theorem 8.1). For $n = 1$, $M_{\mathcal{K}(\mathbb{B}_n)} = H^\infty(\mathbb{B}_n)$ and this is the classical result of von Neumann. However, as we mentioned, for $n \geq 2$ the multiplier space $M_{\mathcal{K}(\mathbb{B}_n)}$ is strictly smaller than $H^\infty(\mathbb{B}_n)$.

Chen [20] has shown that the Drury-Arveson Hardy space $\mathcal{K}(\mathbb{B}_n) = H_n^2$ is isomorphic to the Besov-Sobolev space $B_2^{1/2}(\mathbb{B}_n)$ which can be characterized as consisting of those holomorphic functions $\sum_k a_k z^k$ in the ball with coefficients a_k satisfying

$$\sum_k |a_k|^2 \frac{|k|^{n-1}(n-1)!k!}{(n-1+|k|)!} < \infty.$$

Indeed, the coefficient multipliers in the two previous conditions are easily seen to be comparable for $k > 0$. The comparability of the multiplier norms follows:

$$\|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}} \approx \|f\|_{M_{B_2^{1/2}(\mathbb{B}_n)}}.$$

Hence using Theorem 34, i.e. (8), and Theorem 2 we can give explicit estimates for the function norm in Drury's result. Note however that we only have equivalence of the Hilbert space norms and multiplier space norms, not equality, and that distinction persists in, for instance, the theorem which follows.

Theorem 4 *For any* $m > \frac{n-1}{2}$ *set* $d\mu_f^m(z) = \left|f^{(m)}(z)\right|^2 \left(1 - |z|^2\right)^{2m-n} dz$.

We have

$$\sup_{\substack{A \text{ an} \\ n\text{-contraction}}} \|f(A)\| \approx \|f\|_\infty + \sup_{\alpha\in\mathcal{T}_n} \sqrt{2^{d(\alpha)} I^*\mu_f^m(\alpha)} \tag{12}$$
$$+ \sup_{\alpha\in\mathcal{T}_n} \sqrt{\frac{1}{I^*\mu_f^m(\alpha)} \sum_{k\geq 0} \sum_{\gamma\geq\alpha} 2^{d(\gamma)-k} \sum_{\delta,\delta'\in\mathcal{G}^{(k)}(\gamma)} I^*\mu_f^m(\delta)\, I^*\mu_f^m(\delta')},$$

for all polynomials f on $\mathbb{C}^n$.

The right side of (12) can of course be transported onto the ball using that $\cup_{\beta\geq\alpha} K_\beta$ is an appropriate nonisotropic tent in $\mathbb{B}_n$, and that $2^{-d(\alpha)} \approx 1 - |z|^2$ for $z \in K_\alpha$.

In passing we mention that, inspired partly by the work of Arveson in [12], the space H_n^2 plays a substantial role in modern operator theory. For more recent work see for instance, [3], [13], and [24] .

2.3.4 Carleson measures for Hilbert spaces with a complete *N-P* kernel

The *universal* complete Nevanlinna-Pick property of the Drury-Arveson space H_n^2 allows us to use our description of Carleson measures for H_n^2 to describe Carleson measures for certain other Hilbert spaces. In [2], Agler and M^{c}Carthy consider Hilbert spaces with a complete Nevanlinna-Pick kernel $k(x,y)$. We recall their setup, keeping in mind the classical model of the Szegö kernel $k(x,y) = \frac{1}{1-\overline{x}y}$ on the unit disc $\mathbb{B}_1$. Let X be an infinite set and $k(x,y)$ be a positive definite kernel function on X, i.e. for all finite subsets $\{x_i\}_{i=1}^m$of X,

$$\sum_{i,j=1}^m a_i\overline{a_j} k(x_i,x_j) \geq 0 \text{ with equality } \Leftrightarrow \text{ all } a_i = 0.$$

Denote by $\mathcal{H}_k$ the Hilbert space obtained by completing the space of finite linear combinations of k_{x_i}'s, where $k_x(\cdot) = k(x,\cdot)$, with respect to the inner product

$$\left\langle \sum_{i=1}^m a_i k_{x_i}, \sum_{j=1}^m b_j k_{y_j} \right\rangle = \sum_{i,j=1}^m a_i\overline{b_j} k(x_i,y_j).$$

The kernel k is called a *complete Nevanlinna-Pick kernel* if the solvability of the matrix-valued Nevanlinna-Pick problem is characterized by the contractivity of a certain family of adjoint operators $R_{x,\Lambda}$ (we refer to [2], [3] for an explanation of this generalization of the classical Pick condition).

Let $\mathbb{B}_n$ be the open unit ball in n-dimensional Hilbert space ℓ_n^2; for $n = \infty$, $\ell_\infty^2 = \ell^2(\mathbb{Z}^+)$. For $x, y \in \mathbb{B}_n$ set $a_n(x,y) = \frac{1}{1-\langle y,x\rangle}$ and denote the Hilbert space $\mathcal{H}_{a_n}$ by H_n^2 (so that $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$ when n is finite). Theorem 4.2 of [2] shows

that if k is an irreducible kernel on X, and if for some fixed point $x_0 \in X$, the Hermitian form

$$F(x,y) = 1 - \frac{k(x,x_0)\,k(x_0,y)}{k(x,y)\,k(x_0,y_0)}$$

has rank n, then k is a complete Nevanlinna-Pick kernel if and only if there is an injective function $f : X \to \mathbb{B}_n$ and a nowhere vanishing function δ on X such that

$$k(x,y) = \overline{\delta(x)}\delta(y)\,a_n(f(x), f(y)) = \frac{\overline{\delta(x)}\delta(y)}{1 - \langle f(y), f(x)\rangle}.$$

Moreover, if this happens, then the map $k_x \to \overline{\delta(x)}\,(a_n)_{f(x)}$ extends to an isometric linear embedding T of $\mathcal{H}_k$ into H^2_n. If in addition there is a topology on X so that k is continuous on $X \times X$, then the map f will be a continuous embedding of X into $\mathbb{B}_n$. If X has holomorphic structure and the k_x are holomorphic then f will be holomorphic.

For the remainder of this subsubsection we will assume that X is a topological space and that the kernel function k is continuous on $X \times X$.

In that context we can define a Carleson measure for $\mathcal{H}_k$ to be a positive Borel measure on X for which we have the embedding

$$\int_X |h(x)|^2\, d\mu(x) \le C\,\|h\|^2_{\mathcal{H}_k}, \qquad h \in \mathcal{H}_k, \tag{13}$$

with the standard definition of the Carleson norm. We can now use the description of the Carleson measure norm for $H^2_n = B_2^{1/2}(\mathbb{B}_n)$, given in (7) or in (8) if n is finite and by (4) in any case, to give a necessary and sufficient condition for μ defined on X to be a Carleson measure for $\mathcal{H}_k$. To see this, consider first the case where the Hermitian form F above has finite rank (F is positive semi-definite if k is a complete Nevanlinna-Pick kernel by Theorem 2.1 in [2]). Denote by $f_*\nu$ the pushforward of a Borel measure ν on X under the continuous map f. If μ is a positive Borel measure on X then μ is $\mathcal{H}_k$-Carleson, i.e. (13), if and only if the measure $\mu^\natural = f_*\left(|\delta|^2\,\mu\right)$ is H^2_n-Carleson, i.e.

$$\int_{\mathbb{B}_n} |G|^2\, d\mu^\natural \le C\,\|G\|^2_{B_2^{1/2}(\mathbb{B}_n)}, \qquad G \in B_2^{1/2}(\mathbb{B}_n). \tag{14}$$

Indeed, the functions $h = \sum_{i=1}^m c_i k_{x_i}$ are dense in $\mathcal{H}_k$. Setting $H = Th =$

$\sum_{i=1}^{m} c_i \overline{\delta(x_i)}\,(a_n)_{f(x_i)}$ we have:

$$\begin{aligned}
\|h\|_{\mathcal{H}_k}^2 &= \left\langle \sum_{i=1}^{m} c_i k_{x_i}, \sum_{i=1}^{m} c_i k_{x_i} \right\rangle_{\mathcal{H}_k} = \sum_{i,j=1}^{m} c_i \overline{c_j} k(x_i, x_j) \\
&= \sum_{i,j=1}^{m} c_i \overline{c_j} \overline{\delta(x_i)} \delta(x_j)\, a_n(f(x_i), f(x_j)) \\
&= \left\langle \sum_{i=1}^{m} c_i \overline{\delta(x_i)}\,(a_n)_{f(x_i)}, \sum_{i=1}^{m} c_i \overline{\delta(x_i)}\,(a_n)_{f(x_i)} \right\rangle_{H_n^2} \\
&= \|H\|_{H_n^2}^2 .
\end{aligned}$$

Also, the change of variable f yields

$$\begin{aligned}
\int_X |h(y)|^2 \, d\mu(y) &= \int_X \left| \sum_{i=1}^{m} c_i k(x_i, y) \right|^2 d\mu(y) \\
&= \int_X \left| \sum_{i=1}^{m} c_i \overline{\delta(x_i)} a_n(f(x_i), f(y)) \right|^2 |\delta(y)|^2 \, d\mu(y) \\
&= \int_{f(X)} |H|^2 \, d\mu^\natural = \int_{\mathbb{B}_n} |H|^2 \, d\mu^\natural,
\end{aligned}$$

and it follows immediately that (14) implies (13).

For the converse, we observe that if $G \in H_n^2 = B_2^{1/2}(\mathbb{B}_n)$, then we can write $G = H + J$ where $H \in T(\mathcal{H}_k)$ and J is orthogonal to the closed subspace $T(\mathcal{H}_k)$. Now since J is orthogonal to all functions $\overline{\delta(x)}\,(a_n)_{f(x)}$ with $x \in X$, and since δ is nonvanishing on X, we obtain that J vanishes on the subset $f(X)$ of the ball $\mathbb{B}_n$. Since $\mu^\natural$ is carried by $f(X)$ and orthogonal projections have norm 1, we then have with $H = Th$,

$$\int_{\mathbb{B}_n} |G|^2 \, d\mu^\natural = \int_{\mathbb{B}_n} |H|^2 \, d\mu^\natural = \int_X |h|^2 \, d\mu,$$

and

$$\|h\|_{\mathcal{H}_k} = \|H\|_{H_n^2} \leq \|G\|_{H_n^2} .$$

It follows immediately that (13) implies (14).

We now extend the above characterization to the case of infinite rank. We first characterize Carleson measures on H_∞^2 as follows. Given a finite dimensional subspace L of $\mathbb{C}^\infty$, let P_L denote orthogonal projection onto L and set $\mathbb{B}_L = \mathbb{B}_\infty \cap L$, which we identify with the complex ball $\mathbb{B}_n$, $n = \dim L$. We say that a positive measure ν on $\mathbb{B}_L$ is $H_n^2(\mathbb{B}_L)$-Carleson if, when viewed as a measure on $\mathbb{B}_n$, $n = \dim L$, it is $H_n^2(\mathbb{B}_n)$-Carleson.

Lemma 5 *A positive Borel measure ν on $\mathbb{B}_\infty$ is H_∞^2-Carleson if and only if $(P_L)_* \nu$ is uniformly $H_n^2(\mathbb{B}_L)$-Carleson, $n = \dim L$, for all finite-dimensional subspaces L of $\mathbb{C}^\infty$.*

Proof. Suppose that $(P_L)_* \nu$ is uniformly $H_n^2(\mathbb{B}_L)$-Carleson for all finite-dimensional subspaces L of $\mathbb{C}^\infty$, $n = \dim L$. Let

$$g(z) = \sum_{i=1}^{m} c_i a_\infty(w_i, z) = \sum_{i=1}^{m} c_i \frac{1}{1 - \langle z, w_i \rangle} \tag{15}$$

for a finite sequence $\{w_i\}_{i=1}^m \subset \mathbb{B}_\infty$ (such functions are dense in H_∞^2). If we let L be the linear span of $\{w_i\}_{i=1}^m$ in $\mathbb{C}^\infty$, then since $g(P_L z) = g(z)$, we can view g as a function on both $\mathbb{B}_\infty$ and $\mathbb{B}_L$, and from our hypothesis we have

$$\int_{\mathbb{B}_\infty} |g|^2 \, d\nu = \int_{\mathbb{B}_L} |g|^2 \, d(P_L)_* \nu \leq C \|g\|^2_{H_n^2(\mathbb{B}_L)} = C \|g\|^2_{H_\infty^2}, \tag{16}$$

with a constant C independent of g. Since such functions g are dense in H_∞^2, we conclude that ν is H_∞^2-Carleson. Conversely, given a subspace L and a measure ν that is H_∞^2-Carleson, functions of the form (15) with $\{w_i\}_{i=1}^m \subset \mathbb{B}_L$ are dense in $H_n^2(\mathbb{B}_L)$ and so (16) shows that $(P_L)_* \nu$ is a $H_n^2(\mathbb{B}_L)$-Carleson measure on $\mathbb{B}_L$ with constant C independent of L, $n = \dim L$.

The above lemma together with Lemma 24 below now yields the following characterization of Carleson measures on any Hilbert space $\mathcal{H}_k$ with a complete continuous irreducible Nevanlinna-Pick kernel k. Note that the irreducibility assumption on k can be removed using Lemma 1.1 of [2].

Theorem 6 *With notation as above let k be a complete continuous irreducible Nevanlinna-Pick kernel on a set X and rank $(F) = n$.*

If $n < \infty$ then a positive measure μ on X is $\mathcal{H}_k$-Carleson if and only if $\mu^\natural = f_*(|\delta|^2 \mu)$ is $B_2^{1/2}(\mathbb{B}_n)$-Carleson. That will hold if and only if $\mu^\natural$ satisfies (4) or, equivalently, (6) and (7).

For $n = \infty$, for each finite dimensional subspace L of $\mathbb{C}^\infty$ set

$$\mu_L = (P_L)_* f_*(|\delta|^2 \mu) = (P_L \circ f)_* (|\delta|^2 \mu).$$

A measure μ on X is $\mathcal{H}_k$-Carleson if and only if there is a positive constant C such that for all L

$$\|\mu_L\|_{Carleson} \leq C,$$

Here $\|\nu\|_{Carleson}$ denotes the norm of the embedding $H^2_{\dim L}(\mathbb{B}_L) \subset L^2(\nu)$. This holds if and only if (4) holds (with $\mathbb{B}_n$ taking the role of $\mathbb{B}_L$) uniformly in L.

Because the comparability constants implicit in our proof of (8) depend on dimension we cannot use the right side of (8) in place of $\|\mu_L\|_{Carleson}$ above.

2.3.5 Measures supported on embedded two-manifolds

In the previous discussion we began with a set Ω and kernel function k which satisfied conditions which insured that k could be obtained through a function f mapping Ω into some $\mathbb{B}_n$. Alternatively we can start the analysis with Ω and f.

Given a set Ω and an injective map f of Ω into $\mathbb{B}_n$ set $k(x,y) = a_n(f(x), f(y))$. These kernels generate a Hilbert space $\mathcal{H}_k$ with a complete Nevanlinna-Pick kernel and the previous theorem describes the Carleson measures of $\mathcal{H}_k$. During that proof we also showed that the map T which takes $k(x,\cdot)$ to $a_n(f(x),\cdot)$ extends to an isometric isomorphism of $\mathcal{H}_k$ to the closed span of $\left\{(a_n)_{f(x)} : x \in \Omega\right\}$ in H^2_n. The orthogonal complement of that set is $V_{f(\Omega)}$, the subspace of H^2_n consisting of functions which vanish on $f(\Omega)$. We have

$$\begin{aligned} T(\mathcal{H}_k) &= \text{closed span of } \left\{(a_n)_{f(x)} : x \in \Omega\right\} \\ &= \{h \in H^2_n : h(f(x)) = 0 \ \forall x \in \Omega\}^\perp \\ &= \left(V_{f(\Omega)}\right)^\perp = H^2_n / V_{f(\Omega)}. \end{aligned} \tag{17}$$

The quotient $H^2_n/V_{f(\Omega)}$ can be regarded as a space of functions on $f(\Omega)$ normed by the quotient norm. That space is isometrically isomorphic to $\mathcal{H}_k$ under the mapping which takes $[h]$ in $H^2_n/V_{f(\Omega)}$ to $h \circ f$ in $\mathcal{H}_k$.

We now investigate such embeddings for simple Ω. The L^2 Sobolev space on $[0,1]$ is an example with 1-dimensional Ω. However for this space, and similar 1-dimensional examples, the Carleson measure theory is trivial; a measure is a Carleson measure if and only if it has finite mass. This is reflected in the fact that the associated mapping f of $[0,1]$ into $\mathbb{B}_\infty$ maps the interval into a *proper* sub-ball. (The mappings f associated with this and similar examples are described in the final section of [13].)

Suppose Ω is a bounded domain in the plane and $\partial\Omega$ consists of finitely many smooth curves. (We leave to the reader the straightforward extension to nonplanar domains.) Let f be a nonsingular C^1 embedding of Ω into $\mathbb{B}_n$; $\mathcal{S} = f(\Omega)$. Suppose f extends to a C^1 map of $\bar{\Omega}$ into $\overline{\mathbb{B}_n}$ with $\Gamma = \partial\bar{\mathcal{S}} = f(\partial\bar{\Omega}) \subset \partial\overline{\mathbb{B}_n}$. We will say $\mathcal{S}$ meets the boundary transversally if

$$\operatorname{Re}\langle f'(x)\,\mathbf{n}, f(x)\rangle \neq 0, \qquad x \in \partial\bar{\Omega}, \tag{18}$$

where $\mathbf{n}$ denotes the unit outward normal vector to $\partial\bar{\Omega}$, and $f(x)$ is of course the unit outward normal vector to $\partial\mathbb{B}_n$. In order to discuss various geometric notions of contact at the boundary, we also introduce the unit tangent vector $\mathbf{T}$ to $\partial\bar{\Omega}$ that points in the positive direction, i.e. $\mathbf{T} = i\mathbf{n}$ if the tangent space to $\mathbb{R}^2$ is identified with the complex plane in the usual way. Since the vector $f'(x)\,\mathbf{T}$ is tangent to Γ, we always have

$$\operatorname{Re}\langle f'(x)\,\mathbf{T}, f(x)\rangle = 0, \ x \in \partial\bar{\Omega}.$$

It may also hold that the curve Γ is a complex tangential curve, that is, its tangent lies in the complex tangential tangent direction. This means that the tangent to Γ is perpendicular to the tangential slice direction $if(x)$, i.e. $\operatorname{Re}\langle f'(x)\,\mathbf{T}, if(x)\rangle = 0$ for all $x \in \partial\bar{\Omega}$, or equivalently

$$\operatorname{Im}\langle f'(x)\,\mathbf{T}, f(x)\rangle = 0, \quad x \in \partial\bar{\Omega}. \tag{19}$$

We will say that at the boundary $\mathcal{S}$ is perpendicular to the tangential slice direction and it meets the boundary in the *complex tangential directions*. At the other extreme it may be that $\mathcal{S}$ satisfies (18) and meets the boundary *transverse* to the complex tangential directions, i.e. $f'(x)\,\mathbf{T}$, the tangent to Γ, always has a component in the direction $if(x)$;

$$\operatorname{Im}\langle f'(x)\,\mathbf{T}, f(x)\rangle = \operatorname{Re}\langle f'(x)\,\mathbf{T}, if(x)\rangle \neq 0, \quad x \in \partial\bar{\Omega}, \tag{20}$$

In particular this applies to a *holomorphic* curve, i.e. $\Omega \subset \mathbb{C}$ and f is holomorphic, that satisfies (18) since then we have that $f'(z)$ is *complex* linear and

$$\begin{aligned}\operatorname{Im}\langle f'(z)\,\mathbf{T}, f(z)\rangle &= \operatorname{Im}\langle f'(z)\,i\mathbf{n}, f(z)\rangle = \operatorname{Im} i\langle f'(z)\,\mathbf{n}, f(z)\rangle \\ &= \operatorname{Re}\langle f'(z)\,\mathbf{n}, f(z)\rangle \neq 0, \quad z \in \partial\bar{\Omega}.\end{aligned} \tag{21}$$

Suppose that μ is a positive measure supported on $\mathcal{S}$ that is transverse at the boundary. We will show that if we have additional geometric information about the embedding geometry then the condition for μ to be a Carleson measure for H^2_n can be simplified. Also, as indicated in the previous subsection, this description can be pulled back to give a description of measures on Ω which are Carleson measures for $\mathcal{H}_k$. More precisely we will show that if $\mathcal{S}$ meets the boundary in the complex tangential directions then μ is H^2_n-Carleson if and only if μ satisfies the tree condition (5). On the other hand we show that if $\mathcal{S}$ meets the boundary transverse to the complex tangential directions then μ is H^2_n-Carleson if and only if μ satisfies the simple condition (6). Finally we will show that if f extends continuously but not differentiably to $\partial\bar{\Omega}$ then more complicated situations arise.

To prove these results we use the refined tree structure described in Subsubsection 4.2.1. It is convenient to begin the analysis with the second of the two cases.

$\mathcal{S}$ meets the boundary transverse to the complex tangential directions By Theorem 34, it is enough to show that when $\mathcal{S}$ satisfies (18) and (20), and μ is supported on $\mathcal{S}$ and satisfies the simple condition (6) then for some $\varepsilon > 0$ the ε-split tree condition (9) is satisfied. The transversality hypothesis on $\mathcal{S}$ will permit us to establish a geometric inequality of the following form:

$$d^*([\alpha],[\beta]) \leq d(\alpha,\beta) - \log_2 \frac{1}{|\alpha-\beta|} + c, \qquad \text{when } \mathcal{S}\cap K_\alpha \neq \phi,\ \mathcal{S}\cap K_\beta \neq \phi,$$

at least for $\alpha, \beta \in \mathcal{T}_n$ with $d(\alpha) \approx d(\beta)$ sufficiently large. This in turn will show that the left side of the ε-split tree condition (9) vanishes for ε small enough and $d(\alpha)$ large enough, in fact $0 < \varepsilon < 1/4$ will suffice.

Denote by $P_z w$ the projection of w onto the slice S_z. Suppose that $\mathcal{S}$ satisfies (18) and (20) and fix $z, w \in \mathcal{S}\cap\mathbb{B}_n$ with $1-|z| \approx 1-|w|$, where for the remainder of this subsection the symbol $\approx$ means that the error is small compared to $|z-w|$ times the quantity $\inf_{x\in\partial\Omega} |\operatorname{Im}\langle f'(x)\,\mathbf{T}, f(x)\rangle|$ appearing in (20). Then for $1-|z|$ small enough and $|z-w| \geq c(1-|z|)$, we have

$$|z - P_z w| \geq c\,|w - P_z w|. \tag{22}$$

Indeed, if $z = f(u)$ and $w = f(v)$, then using $f \in C^1\left(\overline{\Omega}\right)$ with (18) and (20) we obtain $c|z-w| \leq |u-v| \leq C|z-w|$ and

$$z - w = f(u) - f(v) \approx f'(u)(u-v).$$

Now let $x \in \partial\Omega$ be closest to u. Using that $u - v \approx \mathbf{T}|u-v|$ we then have

$$f'(u)(u-v) \approx f'(x)(u-v) \approx f'(x)\,\mathbf{T}|u-v|.$$

Since $|z-w| \geq c(1-|z|)$, we also have $f(x) \approx f(u) = z$, and altogether then (20) yields

$$|\mathrm{Im}\,\langle z-w, z\rangle| \approx |\mathrm{Im}\,\langle f'(x)\,\mathbf{T}, f(x)\rangle|\,|u-v| \geq c|u-v| \geq c|z-w|.$$

Thus we obtain (22) as follows:

$$\begin{aligned} |z - P_z w| &= \left|z - \frac{\langle w, z\rangle}{\langle z, z\rangle} z\right| = \frac{1}{|z|}\,|\langle z, z\rangle - \langle w, z\rangle| = \frac{1}{|z|}\,|\langle z-w, z\rangle| \\ &\geq |\mathrm{Im}\,\langle z-w, z\rangle| \geq c|w-z| \geq c|w - P_z w|. \end{aligned}$$

For $x, y \in \mathbb{B}_n$, define $d(x,y)$ to be the corresponding distance in the Bergman tree $\mathcal{T}_n$, i.e. $d(x,y) = d(\alpha,\beta)$ where $x \in K_\alpha$ and $y \in K_\beta$, and $d([x],[y])$ to be the corresponding distance in the ring tree $\mathcal{R}_n$. Recalling that $1-|z| \approx 1-|w|$, and using $A \asymp B$ to mean that $A - B$ is bounded

$$d^*([z],[w]) \asymp d^*([P_z w],[w]) \asymp \log_{\sqrt{2}} \frac{|w - P_z w|}{\sqrt{1-|z|}} = \log_2 \frac{|w - P_z w|^2}{1-|z|}, \tag{23}$$

$$d(z,w) \geq \max\{d([z],[w]), d(z, P_z w)\} \tag{24}$$

$$\geq \max\left\{\log_2 \frac{|w - P_z w|^2}{1-|z|}, \log_2 \frac{|z - P_z w|}{1-|z|}\right\} - c. \tag{25}$$

Combined with (22) this yields

$$\begin{aligned} d^*([z],[w]) &\leq \log_2 \frac{|w - P_z w|^2}{1-|z|} + C = \log_2 \frac{|w - P_z w|}{1-|z|} + \log_2 |w - P_z w| + C \\ &\leq \log_2 \frac{|z - P_z w|}{1-|z|} + \log_2 |w - z| + C \\ &\leq d(z,w) - \log_2 \frac{1}{|w-z|} + C. \end{aligned}$$

Using

$$\begin{aligned} d(z,w) &= d(z) + d(w) - 2d(z \wedge w), \\ d^*([z],[w]) &= d([z]) + d([w]) - 2d^*([z] \wedge [w]), \\ d(z) &= d([z]), \end{aligned}$$

together with $d(z) \asymp d(w)$, we obtain

$$d(z \wedge w) - d^*([z] \wedge [w]) = \frac{1}{2}\left[d^*([z],[w]) - d(z,w)\right] \leq \frac{1}{2}\left[C - \log_2 \frac{1}{|w-z|}\right], \tag{26}$$

for $z, w \in \mathcal{S} \cap \partial\mathbb{B}_n$ with $1 - |z| \approx 1 - |w|$ sufficiently small.

Now let $\alpha, \gamma, \delta, \delta'$ and k be as in the left side of the split tree condition (7) with $K_\delta \cap \mathcal{S} \neq \phi$ and $K_{\delta'} \cap \mathcal{S} \neq \phi$. Thus $\delta \wedge \delta' = \gamma$, $d(\delta) = d(\delta') = d(\gamma) + k + 2$, $\left[A^2\delta\right] = \left[A^2\delta'\right]$ and $d^*([\delta],[\delta']) = 4$. Clearly $|\delta - \delta'| \leq 2^{-\frac{1}{2}d(\gamma)}$ since $\delta, \delta' \geq \gamma$. On the other hand (26) yields

$$d(\gamma) - (d(\gamma) + k) \leq \frac{1}{2}\left[C - \log_2 \frac{1}{|\delta - \delta'|}\right],$$

or $|\delta - \delta'| \geq c2^{-2k}$. Combining these two inequalities for $|\delta - \delta'|$ yields

$$k \geq \frac{1}{4} d(\gamma) - C.$$

Thus the ε-split tree condition (9) for a measure μ supported on $\mathcal{S}$ is vacuous (i.e. the left side vanishes) if $0 < \varepsilon < \frac{1}{4}$ and $\alpha \in \mathcal{T}_n$ is restricted to $d(\alpha)$ large enough. Note that we used only the following consequence of our hypotheses (18) and (20): there are positive constants C, ε, δ such that $\mathcal{S}$ is a subset of $\mathbb{B}_n$ satisfying

$$|x - P_x y| \geq \varepsilon |y - P_x y|, \quad x, y \in \mathcal{S}, \tag{27}$$

whenever $|x| = |y|$, $|x - y| \geq C(1 - |x|)$ and $1 - |x| < \delta$. We have thus proved the following proposition.

Proposition 7 *Suppose $\mathcal{S}$ is a C^1 surface that meets $\partial\mathbb{B}_n$ transversely, i.e. (18) holds, and suppose further that the curve of intersection Γ is transverse to the complex tangential directions, i.e. (20) holds. In particular, $\mathcal{S}$ could be a holomorphic curve embedded in $\mathbb{B}_n$ that is transverse at the boundary $\partial\mathbb{B}_n$.* More generally, *suppose there are positive constants C, ε, δ such that $\mathcal{S}$ is a subset of $\mathbb{B}_n$ satisfying (27) whenever $|x| = |y|$, $|x - y| \geq C(1 - |x|)$ and $1 - |x| < \delta$. Let μ be a positive measure supported on $\mathcal{S}$. Then μ is H^2_n-Carleson if and only if μ satisfies the simple condition (6).*

Corollary 8 *Suppose that $\mathcal{S} = f(\Omega)$ is a C^1 surface that meets the boundary $\partial\mathbb{B}_n$ transversely and that the curve of intersection Γ is transverse to the complex tangential directions. Let $\mathcal{H}_k$ denote the Hilbert space generated by the kernels $k(z,w) = a_n(f(z), f(w))$, $z, w \in \Omega$. Then the Carleson measures for $\mathcal{H}_k$ are characterized by the simple condition (36). In particular this applies to a Riemann surface S and a C^1 embedding f of $\bar{S}$ into $\overline{\mathbb{B}_n}$, holomorphic on S, with $f\left(\partial\bar{S}\right) \subset \partial\mathbb{B}_n$ so that $\mathcal{S} = f(S)$ is transverse at the boundary.*

$\mathcal{S}$ meets the boundary in the complex tangential directions We now suppose $\mathcal{S} = f(\Omega)$ meets the boundary transversely and in the *complex tangential directions*, i.e. $\langle f'(x)\,\mathbf{T}, f(x)\rangle = 0$ for $x \in \partial\bar{\Omega}$. It follows from (2.4) of [5] that

$$1 - \langle f(x), f(x+\delta\mathbf{T})\rangle = \delta^2 \frac{|f'(x)|^2}{2} + o\left(\delta^2\right), \qquad \text{for } x \in \partial\bar{\Omega}, \text{ as } \delta \to 0,$$

where by $x+\delta\mathbf{T}$ we mean the point in $\partial\bar{\Omega}$ that is obtained by flowing along $\partial\bar{\Omega}$ from x a distance δ in the direction of $\mathbf{T}$. From this we obtain

$$|z - P_z w| \leq C\,|w - P_z w|^2 \tag{28}$$

for $z, w \in \mathcal{S} \cap \mathbb{B}_n$ with $1-|z| \approx 1-|w|$ sufficiently small, and $|z-w| \geq c\,(1-|z|)$. Then we obtain from (23) that for such z, w we have

$$d^*([z],[w]) \asymp d(z,w).$$

Hence for μ supported on $\mathcal{S}$, the operator T_μ in (74) below satisfies

$$T_\mu g(\alpha) \approx \sum_{\beta \in \mathcal{T}_n} 2^{d(\alpha\wedge\beta)} g(\beta)\,\mu(\beta), \qquad \alpha \in \mathcal{T}_n,$$

whose boundedness on $\ell^2(\mu)$ is equivalent, by Theorem 23, to the tree condition (5) with $\sigma = 1/2$ i.e. (6). Thus Theorem 30 completes the proof of the following proposition (once we note that if the simple condition holds for a fixed Bergman tree then it holds uniformly for all unitary rotations as well).

Proposition 9 *Suppose that $\mathcal{S}$ is a real 2-manifold embedded in the ball $\mathbb{B}_n$ that meets the boundary transversely and in the complex tangential directions, i.e. both (18) and (19) hold.* More generally, *suppose there are positive constants C, c, δ such that $\mathcal{S}$ is a subset of $\mathbb{B}_n$ satisfying (28) whenever $|x| = |y|$, $|x-y| \geq c\,(1-|x|)$ and $1-|x| < \delta$. Let μ be a positive measure supported on $\mathcal{S}$. Then μ is H^2_n-Carleson if and only if μ satisfies the tree condition (5).*

Remark 10 *This proposition generalizes easily to the case where $\mathcal{S} = f(\Omega)$, $\Omega \subset \mathbb{R}^k$, is a real k-manifold embedded in the ball $\mathbb{B}_n$ that meets the boundary transversely and in the complex tangential directions, i.e.*

$$\langle f'(x)\,\mathbf{T}, f(x)\rangle = 0, \quad x \in \partial\bar{\Omega},$$

for all tangent vectors $\mathbf{T}$ to $\partial\bar{\Omega}$ at x.

For an example of such an embedding let $\Omega = \mathbb{B}_1$ with coordinate $z = x+iy$ and define a mapping into $\mathbb{B}_2$ by $f(z) = (x,y)$. The space $\mathcal{H}_k$ is the Hilbert space of functions on the unit disk with reproducing kernel $k(z,w) = \frac{1}{1-\mathrm{Re}(\bar{z}w)}$. The sublevel sets of this kernel are intersections of the disk with halfplanes and testing against these kernel functions quickly shows that the classic Carleson condition (36) does not describe the Carleson measures for this space. However

the previous proposition together with Theorem 6 gives a description of those measures which turn out to form a subset of the classical Carleson measures. We now provide the details.

Pulling back the kube decomposition from $\mathbb{B}_2$ will give a kube decomposition of $\mathbb{B}_1$ and a tree structure on that set of kubes. However this structure will not be the familiar one from, for instance, Hardy space theory or from [9]. The familiar structure is the following. We define a set of kubes on $\mathbb{B}_1$ by splitting the disk at radii $r_n = 1 - 2^{-n}$ and splitting each ring $\{r_n < |z| \leq r_{n+1}\}$ into 2^n congruent kubes with radial cuts. The tree structure, $\mathcal{T}$, on this set of kubes is described by declaring that α is a successor of β if the radius through the center of α cuts β. On the other hand $\mathcal{F}$, the kube and tree structure pulled back from $\mathbb{B}_2$ by f, is the following. We again split the disk into the same rings and again divide each ring into congruent kubes with radial cuts, but now the number of kubes in that ring is to be $\left[2^{n/2}\right]$. Again the tree structure is described by declaring that α is a successor of β if the radius through the center of α cuts β. Thus the successor sets $S(\alpha) = \cup_{\beta \geq \alpha} \beta$ are approximately rectangles of dimension $2^{-n} \times 2^{-n/2}$, roughly comparable to the complements of sublevel sets of the reproducing kernels for $\mathcal{H}_k$. Note that the number of descendents of a vertex after n generations is quite different for the two trees; in the terminology of [10] $\mathcal{F}$ has tree dimension $1/2$ and $\mathcal{T}$ has tree dimension 1.

We now compare the classes of measures described by (5) for the two different tree structures. We define $B_2^{1/2}(\mathcal{Q})$ on a tree $\mathcal{Q}$ by the norm

$$\|f\|^2_{B_2^{1/2}(\mathcal{Q})} = \sum_{\alpha \in \mathcal{Q}: \alpha \neq o} 2^{-d(\alpha)} \left|f(\alpha) - f(A\alpha)\right|^2 + \left|f(o)\right|^2,$$

for f on the tree $\mathcal{Q}$. Here $A\alpha$ denotes the immediate predecessor of α in the tree $\mathcal{Q}$. We set

$$I_{\mathcal{Q}} f(\alpha) = \sum_{\beta \in \mathcal{Q}: \beta \leq \alpha} f(\beta), \tag{29}$$
$$I^*_{\mathcal{Q}}(g)(\alpha) = \sum_{\beta \in \mathcal{Q}: \beta \geq \alpha} g(\beta).$$

We say that μ is a $B_2^{1/2}(\mathcal{Q})$-Carleson measure on the tree $\mathcal{Q}$ if $B_2^{1/2}(\mathcal{Q})$ imbeds continuously into $L^2_\mu(\mathcal{Q})$, i.e.

$$\sum_{\alpha \in \mathcal{Q}} I_{\mathcal{Q}} f(\alpha)^2 \mu(\alpha) \leq C \sum_{\alpha \in \mathcal{Q}} 2^{-d(\alpha)} f(\alpha)^2, \qquad f \geq 0. \tag{30}$$

We know from [9] that a necessary and sufficient condition for (30) is the discrete tree condition

$$\sum_{\beta \in \mathcal{Q}: \beta \geq \alpha} 2^{d(\beta)} I^*_{\mathcal{Q}} \mu(\beta)^2 \leq C I^*_{\mathcal{Q}} \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{Q}. \tag{$T_{\mathcal{Q}}$}$$

We note a simpler necessary condition for (30)

$$2^{d(\alpha)} I^*_{\mathcal{Q}} \mu(\alpha) \leq C, \tag{$S_{\mathcal{Q}}$}$$

which is obtained using the sum in $(T_{\mathcal{Q}})$ to dominate its largest term. However, condition $(S_{\mathcal{Q}})$ is not in general sufficient for (30) as evidenced by certain Cantor-like measures μ.

These considerations apply when $\mathcal{Q}$ is either of the two trees, $\mathcal{T}$ and $\mathcal{F}$ just described on $\mathbb{B}_1$. However the associated geometries are different; we will refer to conditions associated to $\mathcal{F}$ as "fattened".

Theorem 11 *Let μ be a positive measure on the disk $\mathbb{B}_1$. Then the fattened tree condition $(T_{\mathcal{F}})$ implies the standard tree condition $(T_{\mathcal{T}})$, but not conversely.*

Proof. First we show that the standard tree condition $(T_{\mathcal{T}})$ is not sufficient for the fattened tree condition $(T_{\mathcal{F}})$, in fact not even for the fattened simple condition $(S_{\mathcal{F}})$. For this, let $\rho > -1$ and set

$$d\mu(z) = (1 - |z|)^{\rho}\, dz.$$

Then

$$I_{\mathcal{T}}^{*}\mu(\beta) \approx 2^{-d(\beta)} \int_{1-2^{-d(\beta)}}^{1} (1-r)^{\rho}\, dr \approx 2^{-d(\beta)} \left(2^{-d(\beta)}\right)^{\rho+1} = 2^{-d(\beta)(\rho+2)},$$

and the left side of $(T_{\mathcal{T}})$ satisfies

$$\begin{aligned}
\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} 2^{d(\beta)} I_{\mathcal{T}}^{*}\mu(\beta)^2 &\approx \sum_{\beta\in\mathcal{T}:\beta\geq\alpha} 2^{-d(\beta)(2\rho+3)} \\
&= \sum_{k=d(\alpha)}^{\infty} 2^{k-d(\alpha)} 2^{-k(2\rho+3)} \\
&= 2^{-d(\alpha)} \sum_{k=d(\alpha)}^{\infty} 2^{-k(2\rho+2)} \\
&\approx 2^{-d(\alpha)(2\rho+3)},
\end{aligned}$$

which is dominated by

$$2^{-d(\alpha)(\rho+2)} \approx I_{\mathcal{T}}^{*}\mu(\alpha)$$

if $\rho > -1$. Thus μ satisfies the standard tree condition $(T_{\mathcal{T}})$ for all $\rho > -1$. On the other hand,

$$I_{\mathcal{F}}^{*}\mu(a) \approx 2^{-\frac{d(a)}{2}} \int_{1-2^{-d(a)}}^{1} (1-r)^{\rho}\, dr \approx 2^{-\frac{d(a)}{2}} \left(2^{-d(a)}\right)^{\rho+1} = 2^{-d(a)\left(\rho+\frac{3}{2}\right)},$$

and so the left side of the fattened simple condition $(S_{\mathcal{F}})$ satisfies

$$2^{d(a)} I_{\mathcal{F}}^{*}\mu(a) \approx 2^{d(a)} 2^{-d(a)\left(\rho+\frac{3}{2}\right)} = 2^{-d(a)\left(\rho+\frac{1}{2}\right)},$$

which is unbounded if $\rho < -1/2$. So with $-1 < \rho < -1/2$, $(T_{\mathcal{T}})$ holds but not $(S_{\mathcal{F}})$.

Now we turn to proving that the fattened tree condition $(T_{\mathcal{F}})$ implies the standard tree condition $(T_{\mathcal{T}})$. Decompose the left side of $(T_{\mathcal{T}})$ into the following two pieces:

$$\begin{aligned}\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} 2^{d(\beta)} I_{\mathcal{T}}^{*}\mu\left(\beta\right)^{2} &= \sum_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)\leq 2d(\alpha)} 2^{d(\beta)} I_{\mathcal{T}}^{*}\mu\left(\beta\right)^{2} \\ &+ \sum_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)>2d(\alpha)} 2^{d(\beta)} I_{\mathcal{T}}^{*}\mu\left(\beta\right)^{2} \\ &= A+B.\end{aligned}$$

Now let $a\in\mathcal{F}$ satisfy $d\left(a\right)=2d\left(\alpha\right)$ and

$$\bigcup_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)=2d(\alpha)} K_{\beta}\subset K_{a}, \tag{31}$$

where by K_a for $a\in\mathcal{F}$ we mean the fattened kube in the disk corresponding to a (it is roughly a $2^{-d(\beta)}\times 2^{-\frac{d(\beta)}{2}}$ rectangle - which is $2^{-2d(\alpha)}\times 2^{-d(\alpha)}$ - oriented so that its long side is parallel to the nearby boundary of the disk, and so that its distance from the boundary is about $2^{-d(a)}$). It may be that two such adjacent kubes K_a and $K_{a'}$ are required to cover the left side of (31), but the argument below can be easily modified to accommodate this upon replacing μ by $\mu\chi$ where χ denotes the characteristic function of the successor set $S_{\alpha}=\cup_{\beta\in\mathcal{T}:\beta\geq\alpha}K_{\beta}$ and noting from (30) that if μ satisfies $(T_{\mathcal{F}})$ then so does $\mu\chi$. Then we have

$$\begin{aligned}B &= \sum_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)>2d(\alpha)} 2^{d(\beta)} I_{\mathcal{T}}^{*}\mu\left(\beta\right)^{2} \\ &\leq \sum_{b\in\mathcal{F}:b\geq a} 2^{d(b)} \sum_{\beta\in\mathcal{T}:K_{\beta}\subset K_{b}} I_{\mathcal{T}}^{*}\mu\left(\beta\right)^{2} \\ &\leq \sum_{b\in\mathcal{F}:b\geq a} 2^{d(b)} \left(\sum_{\beta\in\mathcal{T}:K_{\beta}\subset K_{b}} I_{\mathcal{T}}^{*}\mu\left(\beta\right)\right)^{2} \\ &\leq \sum_{b\in\mathcal{F}:b\geq a} 2^{d(b)} I_{\mathcal{F}}^{*}\mu\left(b\right)^{2}.\end{aligned}$$

The fattened tree condition $(T_{\mathcal{F}})$ shows that the final term above is dominated by $CI_{\mathcal{F}}^{*}\mu\left(a\right)$, which is at most $CI_{\mathcal{T}}^{*}\mu\left(\alpha\right)$, and hence we have

$$B\leq CI_{\mathcal{T}}^{*}\mu\left(\alpha\right).$$

To handle term A we write the geodesic in $\mathcal{F}$ consisting of a together with the $d\left(\alpha\right)$ terms immediately preceding a in $\mathcal{F}$ as

$$\left\{a_{d(\alpha)},a_{d(\alpha)+1},...,a_{2d(\alpha)}=a\right\},$$

where $d(a_k) = k$ and $a_k < a_{k+1}$. Then

$$\begin{aligned} A &\leq \sum_{k=d(\alpha)}^{2d(\alpha)} 2^k \sum_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)=k} I_{\mathcal{T}}^{*}\mu(\beta)^2 \\ &\leq \sum_{k=d(\alpha)}^{2d(\alpha)} 2^k \left(\sum_{\beta\in\mathcal{T}:\beta\geq\alpha \text{ and } d(\beta)=k} I_{\mathcal{T}}^{*}\mu(\beta) \right)^2 \\ &\leq \sum_{k=d(\alpha)}^{2d(\alpha)} 2^k I_{\mathcal{F}}^{*}(\chi\mu)(a_k)^2 . \end{aligned}$$

Now for $j \geq 0$, let E_j consist of those integers k in $[d(\alpha), 2d(\alpha)]$ satisfying

$$2^{-j-1} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right) < I_{\mathcal{F}}^{*}(\chi\mu)(a_k) \leq 2^{-j} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right), \tag{32}$$

and provided $E_j \neq \phi$, let $k_j = \max_{E_j} k$ be the largest integer in E_j, so that

$$2^{-j-1} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right) < I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{k_j}\right) \leq 2^{-j} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right). \tag{33}$$

Using (32) and (33), we then have

$$\begin{aligned} A &\leq 2\sum_{j\geq 0} 2^{-j} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right) I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{k_j}\right) \left\{ \sum_{k\in E_j} 2^k \right\} \\ &\leq 4\sum_{j\geq 0} 2^{-j} I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right) \left\{ I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{k_j}\right) 2^{k_j} \right\} \\ &\leq C I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right), \end{aligned}$$

where the last line follows from the fattened simple condition $(S_{\mathcal{F}})$ applied to a_{k_j} since $d\left(a_{k_j}\right) = k_j$. Since

$$I_{\mathcal{F}}^{*}(\chi\mu)\left(a_{d(\alpha)}\right) \leq C I_{\mathcal{T}}^{*}\mu(\alpha),$$

we have altogether,

$$A + B \leq C I_{\mathcal{T}}^{*}\mu(\alpha),$$

which completes the proof that the standard tree condition $(T_{\mathcal{T}})$ holds when the fattened tree condition $(T_{\mathcal{F}})$ holds.

The embedding is Lipschitz continuous to the boundary but not C^1 In the next section we will see that if $\mathbb{B}_1$ is embedded holomorphically in $\mathbb{B}_n$ and the embedding has a transverse C^2 extension that takes $\partial\mathbb{B}_1$ into $\partial\mathbb{B}_n$ then the induced space of functions on the embedded disk is the Hardy space of the disk. The proof is given for finite n but it only uses the fact that the kernel functions on the disk have useful second order Taylor expansions; hence an analog of the result holds if $n = \infty$. We now give an example where the embedding extends

continuously to the boundary but the induced function space on the disk is $B_2^\sigma(\mathbb{B}_1)$ with $0 < \sigma < 1/2$ and not the Hardy space $B_2^{1/2}(\mathbb{B}_1)$. In fact in Subsubsection 2.3.4 we saw that there must be an embedding of the disk into a $\mathbb{B}_n$ so that the induced function space on $\mathbb{B}_1$ is $B_2^\sigma(\mathbb{B}_1)$. Here we write the map explicitly and do certain computations.

Pick and fix σ, $0 < \sigma < 1/2$. We want a map f of $\mathbb{B}_1$ into $\mathbb{B}_\infty$ so that

$$\frac{1}{(1-\bar{x}y)^{2\sigma}} = \frac{1}{\left(1-\overline{f(x)}\cdot f(y)\right)}. \tag{34}$$

Define c_n by

$$1-(1-z)^{2\sigma} = \sum_1^\infty c_n z^n$$

and define $f : \mathbb{B}_1 \to \mathbb{B}_\infty$ by

$$f(z) = \left(\sqrt{c_n}z^n\right)_1^\infty,$$

hence (34) holds.

We know c_n are positive and

$$\begin{aligned} c_n &= \left|\binom{2\sigma}{n}\right| = \frac{2\sigma}{n}\left(1-\frac{2\sigma}{1}\right)...\left(1-\frac{2\sigma}{n-1}\right) \\ &\approx \frac{2\sigma}{n}e^{-\left(\frac{2\sigma}{1}+...+\frac{2\sigma}{n-1}\right)} \approx \frac{2\sigma}{n}e^{-2\sigma\ln n} \approx n^{-1-2\sigma}. \end{aligned}$$

Thus f extends continuously to the boundary but, for $z \in \partial\mathbb{B}_1$, $f'(z)$ fails to be in l^∞ much less l^2. To estimate the behavior of f near the boundary we use the fact that $1-r^n \approx n(1-r)$ for $n \leq \frac{1}{1-r}$ and estimate

$$\begin{aligned} |f(1)-f(r)|^2 &= \sum_{n=1}^\infty \left|\sqrt{c_n}\left(1-r^n\right)\right|^2 \\ &\approx \sum_{n=1}^{\frac{1}{1-r}} n^{-1-2\sigma}\left(1-r^n\right)^2 + \sum_{n=\frac{1}{1-r}}^\infty n^{-1-2\sigma}\left(1-r^n\right)^2 \\ &\approx \sum_{n=1}^{\frac{1}{1-r}} n^{-1-2\sigma}n^2(1-r)^2 + \sum_{n=\frac{1}{1-r}}^\infty n^{-1-2\sigma} \approx (1-r)^{2\sigma}, \end{aligned}$$

so that

$$|f(1)-f(r)| \approx (1-r)^\sigma.$$

Thus f is $Lip\,\sigma$.

Suppose we now take a point x on the positive real axis near the boundary. The image point is $f(x) = \left(\sqrt{c_n}x^n\right)$ and the distance of $f(x)$ to the boundary

is

$$\begin{aligned}
1-\left(\overline{f(x)}\cdot f(x)\right)^{1/2} &= 1-\left(\sum c_n x^{2n}\right)^{1/2} \\
&= 1-\left(1-\left(1-x^2\right)^{2\sigma}\right)^{1/2} \\
&\sim 1-\left(1-\frac{1}{2}(1-x^2)^{2\sigma}\right) \\
&\sim (1-x^2)^{2\sigma}.
\end{aligned}$$

Because f is not differentiable at the boundary our earlier definition of transverse does not apply. However f does fail to be transverse at the boundary in the sense that

$$\frac{dist\left(f(r),\partial\mathbb{B}_\infty\right)}{dist\left(f(r),f(1)\right)}=\frac{1-\left(\overline{f(r)}\cdot f(r)\right)^{1/2}}{|f(1)-f(r)|}\approx\frac{(1-r)^{2\sigma}}{(1-r)^{\sigma}}=(1-r)^{\sigma}$$

is not bounded below as $r\to 1$; as it would be if we had (18)**.**

Now consider Carleson measures. We know that a measure μ on the disk is a Carleson measure for $B_2^{\sigma}\left(\mathbb{B}_1\right)$ if and only if $f_*\mu$ is a Carleson measure for $B_2^{1/2}\left(\mathbb{B}_\infty\right)$. Here we just note that it is straightforward to check that the simple condition SC(σ) for μ corresponds to the SC(1/2) condition for $f_*\mu$. Fix x in the disk, near the boundary. The SC(1/2) condition for $f_*\mu$ states that the μ mass of the set of y for which

$$\left|1-\overline{f(y)}\cdot\frac{f(x)}{\|f(x)\|}\right|\leq 1-\|f(x)\|.$$

is dominated by $C\left(1-\|f(x)\|\right)$. Using the closed form for $\sum c_n z^n$ to evaluate the norms and the inner product and doing a bit of algebra we find that set is the same as the set of y for which

$$\left|\left(1-\frac{x(\overline{y-x})}{1-|x|^2}\right)^{2\sigma}-1\right|\leq 1-\left(1-|x|^2\right)^{2\sigma}.$$

This is in turn equivalent to $|y-x|\leq C(1-|x|)$ which describes a set of y's comparable in size and shape with the set of y for which

$$\left|1-\overline{y}\cdot\frac{x}{|x|}\right|\leq C\left(1-|x|\right).$$

The conclusion now follows from the comparison $1-\|f(x)\|\sim(1-|x|)^{2\sigma}$.

We just studied f using the Euclidean metric for both $\mathbb{B}_1$ and $\mathbb{B}_\infty$. There are other natural metrics in this context. For fixed n, σ we can define the metric δ_σ on $\mathbb{B}_n$ by

$$\begin{aligned}
\delta_\sigma(x,y) &= \sqrt{1-\frac{|k(x,y)|^2}{k(x,x)k(y,y)}} \\
&= \sin(\text{angle}(k(x,\cdot)k(y,\cdot))),
\end{aligned}$$

where $k_\sigma(x,y) = k_\sigma(x,y) = (1 - \overline{x} \cdot y)^{-2\sigma}$ is the reproducing kernel for $B_2^\sigma(\mathbb{B}_n)$. This is a general construction of a metric associated with a reproducing kernel Hilbert space and is related to the themes we have been considering, see Section 9.2 of [3]. For the particular map $f = f_\sigma$ we defined it is a consequence of (34) and the definitions that $f = f_\sigma$ will be an *isometry* from $(\mathbb{B}_1, \delta_\sigma)$ into $(\mathbb{B}_\infty, \delta_{1/2})$.

2.3.6 Hardy spaces on planar domains

Suppose now that $\Omega = R$, a domain in $\mathbb{C}$ with boundary $\Gamma = \partial\bar{R}$ consisting of a finite collection of C^2 curves. Suppose that f is a holomorphic map of R into some $\mathbb{B}_n$, that f extends to a C^1 map of $\bar{R}$ into $\bar{\mathbb{B}}_n$ which takes Γ into $\partial\bar{\mathbb{B}}_n$, which is one to one on Γ, and which satisfies the transversality condition

$$\langle f'(z), f(z)\rangle \neq 0 \text{ for } z \in \Gamma \tag{35}$$

which combines (20) and (18); recall that these two conditions are equivalent for holomorphic embeddings. We denote the space generated by the kernel functions $k(x,y) = a_n(f(x), f(y))$ by $\mathcal{H}_k(R)$. (This is a minor variation on what was described earlier; here we do not require that f be injective on R.)

We want to study the relation between $\mathcal{H}_k(R)$ and the Hardy space of R, $H^2(R)$, which we now define. Let $d\sigma$ be arclength measure on Γ and define $H^2 = H^2(R)$ to be the closure in $L^2(\Gamma, d\sigma)$ of the subspace consisting of restrictions to Γ of functions holomorphic on $\bar{R}$. We refer to [1] and [23] for the basic theory of these spaces. In particular there is a natural isometric identification of H^2 as a space of nontangential boundary values of a certain space of holomorphic functions on R, we also denote that space by H^2. The choice of the measure $d\sigma$ is not canonical but all the standard choices lead to the same space of holomorphic functions on R with equivalent norms. The Carleson measures for H^2 are those described by the classical Carleson condition, measures μ for which there is a constant C so that for all $r > 0$, $z \in \Gamma$

$$\mu(B(z,r) \cap R) \leq Cr. \tag{36}$$

That is, μ satisfies the appropriate version of the simple condition (6). For small positive ε and $z \in \Gamma$ let $\varepsilon(z)$ be the inward pointing normal at z of length ε. Because the norm in H^2 can be computed as

$$\overline{\lim_{\varepsilon \to 0^+}} \int_\Gamma |f(z + \varepsilon(z))|^2 \, d\sigma$$

and those integrals are, in fact, the integration of $|f|^2$ against a measure on R which satisfies (36); we have that H^2 is saturated with respect to its Carleson measures; H^2 consists of exactly those holomorphic functions for which

$$\sup\left\{ \int_R |f|^2 \, d\mu : \mu \in CM(H^2), \ \ \|\mu\|_{Carleson} = 1 \right\} < \infty. \tag{37}$$

(We note in passing that if a Banach space of holomorphic functions B is saturated with respect to its Carleson measures then the multiplier algebra of B

will be H^∞: thus, by the comments following Theorem 2, the Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$ are not saturated for $0 \leq \sigma \leq 1/2$ except for the classical Hardy space, $n = 1, \sigma = 1/2$.)

We saw earlier (Proposition 7) that as a consequence of the transversallity condition the Carleson measures for $\mathcal{H}_k(R)$ are exactly those which satisfy (36). Hence every $f \in \mathcal{H}_k(R)$ satisfies (37) and thus we have a continuous inclusion i

$$i : \mathcal{H}_k(R) \to H^2. \tag{38}$$

To this point we have only used that f is C^1. We will be able to get much more precise information about the relation between $\mathcal{H}_k(R)$ and H^2 if we assume that f is C^2. We now make that assumption.

The prototype for our analysis is the proof by D. Alpay, M. Putinar, and V. Vinnakov [6] that if $R = \mathbb{B}_1$, f is C^2, and if the differential df is nonvanishing, then $\mathcal{H}_k(\mathbb{B}_1) = H^2(\mathbb{B}_1)$; the spaces of functions coincide and the norms are equivalent. This insures that the spaces have the same multipliers. We know from the classical theory that the multiplier algebra of $H^2(\mathbb{B}_1)$ is $H^\infty(\mathbb{B}_1)$ and hence the multiplier algebra of $\mathcal{H}_k(\mathbb{B}_1)$ is also $H^\infty(\mathbb{B}_1)$. Application of the theory of complete Nevanlinna-Pick kernels then gives an interesting consequence; any bounded holomorphic function on $f(\mathbb{B}_1)$ has a holomorphic extension to all $\mathbb{B}_n$ which is bounded and, in fact, is in the multiplier algebra $M_{B_2^{1/2}(\mathbb{B}_n)}$. Indeed, this uses the fact that the multiplier algebra of $\mathcal{H}_k(\mathbb{B}_1)$ is $H^\infty(\mathbb{B}_1)$ as follows. If $h \in H^\infty(f(\mathbb{B}_1))$ with norm 1, then $h \circ f \in H^\infty(\mathbb{B}_1) = M_{\mathcal{H}_k(\mathbb{B}_1)}$ with norm $M < \infty$, and thus the matrices

$$\left[\left(M^2 - h(f(z_i))\,\overline{h(f(z_j))}k(z_i, z_j)\right)\right]_{i,j=1}^m$$

are positive semi-definite for all infinite sequences $\{z_i\}_{i=1}^\infty$ in $\mathbb{B}_1$ and m finite. By the definition of k, this says that the matrices

$$\left[\left(M^2 - h(w_i)\,\overline{h(w_j)}a_n(w_i, w_j)\right)\right]_{i,j=1}^m$$

are positive semi-definite for all infinite sequences $\{w_i\}_{i=1}^\infty$ in $f(\mathbb{B}_1)$ and m finite. Taking $\{w_i\}_{i=1}^\infty$ to be dense in $f(\mathbb{B}_1)$, the Pick property for H_n^2 shows that there is $\varphi \in M_{H_n^2}$ with $\|\varphi\|_{M_{H_n^2}} \leq M$ and that agrees with h on $\{w_i\}_{i=1}^\infty$, hence on $f(\mathbb{B}_1)$ as required. See [6] for details. (In fact there is a minor error in that paper; a nonsingularity hypothesis is needed as shown by the map of $\mathbb{B}_1$into $\mathbb{B}_2$ given by $f(z) = 2^{-1/2}(z^2, z^3)$. For this choice of f the space $\mathcal{H}_k(\mathbb{B}_1)$ will not contain any g with $g'(0) \neq 0$. The hypothesis is needed to insure that the function ϕ^{-1} constructed at the end of Section 3 of [6] has the required properties. Also, the continuity properties of the function L in [6] follow if f is assumed to be C^2.)

We know the inclusion i is bounded, we now turn attention to its adjoint i^* mapping H^2 to $\mathcal{H}_k$. We want to compute the norm $\|i^*g\|_{\mathcal{H}_k}$ for g a finite linear combination of kernel functions. We denote the kernel functions for $\mathcal{H}_k$ by

$k_x = k(x, \cdot)$ and those for the Hardy space H^2 by h_x. It is a direct computation that for any x, $i^* h_x = k_x$. Thus if $g = \sum a_i h_{x_i}$ then $i^*(g) = \sum a_i k_{x_i}$ and

$$\begin{aligned} \|i^* g\|^2_{\mathcal{H}_k} &= \left\langle \textstyle\sum_i a_i k_{x_i}, \sum_j a_j k_{x_j} \right\rangle_{\mathcal{H}_k} \\ &= \textstyle\sum_{i,j} a_i \bar{a}_j \left\langle k_{x_i}, k_{x_j} \right\rangle_{\mathcal{H}_k} = \sum_{i,j} a_i \bar{a}_j k_{x_i}(x_j). \end{aligned}$$

Alternatively, setting $\tilde{S} = ii^*$, we have

$$\begin{aligned} \|i^* g\|^2_{\mathcal{H}_k} &= \langle i^* g, i^* g \rangle_{\mathcal{H}_k} = \left\langle \tilde{S} g, g \right\rangle_{H^2} \\ &= \textstyle\sum_{i,j} a_i \bar{a}_j \left\langle \tilde{S} h_{x_i}, h_{x_j} \right\rangle_{H^2}. \end{aligned}$$

Thus $\tilde{S}$ is a positive operator on H^2 and we know $\tilde{S}$ is bounded because we know i is bounded. We record the consequence

$$\left\langle \tilde{S} h_{x_i}, h_{x_j} \right\rangle_{H^2} = k_{x_i}(x_j). \tag{39}$$

We now give an integral representation of $\tilde{S}$ and using that show that $\tilde{S}$ is a Fredholm operator. For $h \in H^2$ define S by

$$Sh(x) = \int_\Gamma k_\omega(x) h(\omega) d\sigma(\omega), \quad x \in R.$$

In particular, setting $h = h_{x_i}$ we have

$$\begin{aligned} Sh_{x_i}(x) &= \int_\Gamma k_\omega(x)\, h_{x_i}(\omega)\, d\sigma(\omega) \\ &= \left\langle k_{(\cdot)}(x), \overline{h_{x_i}(\cdot)} \right\rangle_{H^2} \\ &= k_{x_i}(x). \end{aligned}$$

In the last equality we used the fact that $k_\omega(x)$ is bounded and conjugate holomorphic in ω and that taking the inner product with $\bar{h}_{x_i}$ evaluates such a function at x_i. Hence we have

$$\left\langle S h_{x_i}, h_{x_j} \right\rangle_{H^2} = k_{x_i}(x_j).$$

Comparing with (39) we conclude that $S = \tilde{S}$. Following [6] we now compare the integration kernel for S with the Cauchy kernel. For $\omega \in \Gamma, \zeta \in R$ we set

$$\begin{aligned} L(\omega, \zeta) = (\omega - \zeta) k_\omega(\zeta) &= \frac{(\omega - \zeta)}{1 - f(\zeta) \cdot \overline{f(\omega)}} \\ &= \frac{(\omega - \zeta)}{(f(\omega) - f(\zeta)) \cdot \overline{f(\omega)}}. \end{aligned}$$

The transversality hypothesis insures that L extends continuously to $\bar{R} \times \bar{R}$ and that for $\omega \in \Gamma$ we have $L(\omega, \omega) = \langle f'(\omega), f(\omega) \rangle^{-1}$, a continuous function that is bounded away from zero. We now write the integration kernel for S as

$$\begin{aligned} k_\omega(\zeta) &= \frac{L(\omega, \zeta)}{\omega - \zeta} \\ &= \frac{L(\omega, \zeta) - L(\omega, \omega)}{\omega - \zeta} + \frac{L(\omega, \omega)}{\omega - \zeta} \\ &= k_{1,\omega}(\zeta) + k_{2,\omega}(\zeta). \end{aligned}$$

This lets us split $S = S_1 + S_2$. The hypothesis that f be C^2 insures that k_1 extends $\bar{R} \times \bar{R}$ with

$$k_{1,\omega}(\omega) = \frac{f''(\omega) \cdot \overline{f(\omega)}}{2\left(f'(\omega) \cdot \overline{f(\omega)}\right)^2},$$

a continuous function, and hence S_1 is compact. Along Γ we can write $dz(\omega) = v(\omega) d\sigma(\omega)$ for a continuous function v which is bounded away from 0. Thus

$$S_2 h(x) = \frac{1}{2\pi i} \int_\Gamma \frac{1}{x - \omega} s(\omega) h(\omega) dz(\omega).$$

with s continuous and bounded away from zero. The operator

$$Ph = \frac{1}{2\pi i} \int_\Gamma \frac{1}{x - \omega} h(\omega) dz(\omega)$$

gives a bounded projection of $L^2(\Gamma, d\sigma)$ onto H^2 [1]. Thus S_2 is a Toeplitz operator with a symbol that is continuous and bounded away from zero. Hence, by the Fredholm theory for Toeplitz operators, S_2 is a Fredholm operator [1]. Hence S, a compact perturbation of S_2, is also Fredholm.

S is a positive Fredholm operator on H^2. Hence $\mathrm{Ker}(S)$ is finite dimensional and $\mathrm{Ran}(S)$ is the closed subspace $\mathrm{Ker}(S)^\perp$. The restriction of S to that closed subspace is an isomorphism of that space; one-to-one, onto, bounded, and bounded below.

We can identify $\mathcal{H}_k$ with $H_n^2 / V_{f(X)}$. In particular if P is any polynomial on $\mathbb{C}^n$ then there is a function in $\mathcal{H}_k$ of the form $\tilde{P} = P \circ f$. Furthermore we have the norm estimate

$$\left\| \tilde{P} \right\|_{\mathcal{H}_k} = \|P\|_{H_n^2 / V_{f(X)}} \leq \|P\|_{H_n^2}.$$

Using this and the fact that the polynomials are dense in H_n^2 we conclude that the set $B_0(R) = \{\tilde{P}\}$ is dense in $\mathcal{H}_k$. Let $A(R)$ be the algebra of functions holomorphic in R which extend continuously to $\bar{R}$, normed by the uniform norm. Because f extends continuously to $\bar{R}$ the set $B_0(R)$ is a subalgebra of $A(R)$. Let $B(R)$ denote the closure of $B_0(R)$ in $A(R)$ and let $\overline{B(R)}$ denote the closure of $B(R)$ (or, equivalently, $B_0(R)$) in H^2.

Suppose now that we have $f \in H^2$ in $\mathrm{Ker}(S)$. We know i is injective hence we must have $i^* f = 0$. Thus, for any $P \in B_0(R)$, $\langle i^* f, P \rangle_{\mathcal{H}_k} = \langle f, iP \rangle_{H^2} = 0$. Hence $\mathrm{Ker}(S) \subset \overline{B(R)}^{\perp}$. When we pass to orthogonal complements and recall that $\mathrm{Ran}(S) = \mathrm{Ker}(S)^{\perp}$ we find that $\mathrm{Ran}(S) \supset \overline{B(R)}$. On the other hand $S = ii^*$ and hence $\mathrm{Ran}(S) \subset \mathrm{Ran}(i)$. We know $B_0(R)$ is $\mathcal{H}_k$-dense in $\mathcal{H}_k$ and that i is continuous. Thus we continue the inclusions with $\mathrm{Ran}(S) \subset \mathrm{Ran}(i) \subset \overline{B(R)}$. Combining these ingredients we have $\mathrm{Ran}(S) = \mathrm{Ran}(i) = \overline{B(R)}$. In particular we have that i is a continuous one-to-one map onto its closed range and hence must be a norm isomorphism of $\mathcal{H}_k(R)$ and $\overline{B(R)}$. In sum we have

Theorem 12 *Suppose f, R, and $\overline{B(R)}$ are as described. Then $\overline{B(R)}$ has finite codimension in H^2 and i is a norm isomorphism between $\mathcal{H}_k(R)$ and $\overline{B(R)}$.*

If

$$\dim(A(R)/B(R)) = s < \infty. \tag{40}$$

then the codimension of $\overline{B(R)}$ in H^2 is s. In particular if $B_0(R)$ is dense in $A(R)$ then i is a norm isomorphism of $\mathcal{H}_k(R)$ onto H^2.

Proof. We have established the first statements. Suppose now that (40) holds. By work of T. Gamelin [25] we have a complete structural description of $B(R)$. The algebra $B(R)$ can be obtained from $A(R)$ by a chain of passages to subalgebras each of codimension one in the previous subalgebra. Each of these steps is of one of two possible forms. One possible step consists of selecting two points x and y of R and passing to the subalgebra of functions which take the same values at x and y. The other possibility is picking a point x in R and passing to the kernel of a point derivation (in the algebra considered) supported at x. In particular, at each step we pass to the kernel of a linear functional which can extended continuously to H^2 and is thus given by an inner product with a $h \in H^2$. (This is because the points x, y were in R, not in Γ.) Thus there are s elements in H^2 such that $B(R) = A(R) \cap (\mathrm{span}\{h_1, ..., h_s\})^{\perp}$. This insures that when we pass to closures in H^2 we will have $\overline{B(R)} = (\mathrm{span}\{h_1, ..., h_s\})^{\perp}$ which has codimension s. ■

Corollary 13 *If R is a domain in $\mathbb{C}$ with boundary consisting of a finite collection of smooth curves then the Hardy space $H^2(R)$ admits an equivalent norm with the property that with the new norm the space is a reproducing kernel Hilbert space with complete N-P kernel.*

Proof. It suffices to find a mapping f of R into some $\mathbb{B}_n$ to which the previous theorem applies and so that $B_0(R)$ is dense in $A(R)$. It is a theorem of Stout [35] that one can find a set of three holomorphic functions $\{f_i\}_{i=1,2,3}$ which separate points, with each f_i having modulus identically one on each boundary component and such that there is no point at which all three functions have vanishing derivative. In the same paper he shows that under these assumptions the polynomials in the f_i are dense in $A(R)$. We now claim that the mapping of z to $f(z) = (f_1(z), f_2(z), f_3(z))/\sqrt{3}$ is a map to which the theorem can be applied. To see that each f_i has the required smoothness note that if one precomposes

with a conformal map ϕ which takes part of the unit disc near 1 to the part of R near a boundary point z then by the reflection principle the composite is real analytic at 1. Hence near z the f_i are as smooth as ϕ and the local smoothness of ϕ is determined by the smoothness of Γ. Also note that, assuming f_i is not constant, $(f_i \circ \phi)'(1) \neq 0$, for otherwise the image of a neighborhood of 1 under the holomorphic map $f_i \circ \phi$ would not stay *inside* the unit disk. In particular, for each nonconstant f_i we have $f_i' \neq 0$ on Γ. To finish we need to verify the transversality condition (35). For $z \in \Gamma$, for the nonconstant f_i, $f_i'(z)\overline{f_i(z)} \neq 0$. We need to insure that if several such terms are added there is no cancellation. That follows from applying the following lemma to each of the f_i and noting that the number α in the lemma is determined by the geometry at the point z but is independent of the function g. ■

Lemma 14 *Suppose R is a domain in $\mathbb{C}$, γ is a C^2 arc forming part of $\partial\bar{R}$ and $z \in \gamma$. There is a real number α so that for any g holomorphic on R with $|g(z)| = 1$ on γ, $g'(z) \neq 0$ and $|g(w)| \leq 1$ on the intersection of R with a neighborhood of z we have $g'(z)\overline{g(z)} = e^{i\alpha}r(g)$ for some positive real number $r(g)$.*

Proof. First consider the case when γ is part of the unit circle near $z = 1$ and locally R is inside the circle. By the reflection principle g extends to a holomorphic function on a neighbourhood of z which insures that the hypotheses about the boundary behavior of g are well formulated. We have $g(1) = \eta$ with $|\eta| = 1$. By conformality and the fact that g takes part of the circle to part of the circle, the linearization of g must map the outward pointing normal at 1 to the outward pointing normal at η. Thus $g'(1) = \eta r$ for some positive r and hence $g'(1)\overline{g(1)} = \eta r\bar{\eta} = r$ as required. For the general case let ϕ be a conformal mapping of the part of the unit disc near 1 to the interior of R near z which takes 1 to z. If γ is C^2 near z then ϕ will be at least C^1 at 1 and thus we can apply the result from the special case to $g \circ \phi$. That gives

$$0 < (g \circ \phi)'(1)\overline{(g \circ \phi)(1)} = g'(\phi(1))\phi'(1)\overline{g(\phi(1))} = g'(z)\phi'(1)\overline{g(z)}$$

and $\arg(g'(z)\overline{g(z)})$ is independent of g as required. ■

Remark 15 *A reason for taking note of this corollary is that, while it is known that the classical Hardy space of the disc does have a complete N-P kernel, the various classically defined norms on the Hardy spaces of multiply connected domains do not have this property. (Actually it is not known if the property always fails; it is known to fail sometimes and there are no known cases until now using the classically defined norms where it holds.) Hence it is interesting that the spaces do carry relatively natural equivalent norms with the property. See [3] for further discussion.*

Also, as in [6], we obtain extension theorems as follows. Suppose now that we are in the situation of the previous theorem and (40) holds. As noted in that proof, we will have $\mathcal{H}_k(R) = V^\perp$ where V is a *finite* dimensional subspace of

$H^2(R)$ and the orthogonality is in $H^2(R)$. In that case the multiplier algebra $\mathcal{H}_k(R)$, $M_{\mathcal{H}_k(R)}$, will be

$$H^\infty(R) \cap \mathcal{H}_k(R) = H^\infty(R) \cap V^\perp = w^*\text{-closure of } B(R) \text{ in } H^\infty(R).$$

The facts that multipliers must be bounded and that $1 \in \mathcal{H}_k(R)$ insure $M_{\mathcal{H}_k(R)}$ is contained in that space. On the other hand if $b \in H^\infty(R)$ then b multiplies $\mathcal{H}_k(R)$ into $H^2(R)$. We then need to know that if b is also in $V^\perp$ and that if $g \in V^\perp$ then bg, which we know to be in $H^2(R)$, is also in $V^\perp$. That is insured by the fact that membership in $V^\perp$ is determined by local conditions which have the form that if two functions satisfy them then so does the product.

The fact that membership in $V^\perp$ is determined locally allows us to have a more intrinsic description of the multipliers. First note that for any function h in $H^\infty(R) \cap \mathcal{H}_k(R)$, the function Th defined on $f(R)$ by $Th(f(z)) = h(z)$ is a function on $f(R)$ which can be obtained by restricting a function in H^2_n to $f(R)$. This insures that given $z \in f(R)$ there is a neighbourhood $V_z \subset \mathbb{B}_n$ and a holomorphic function h^*_z defined on V_z such that $h^*_z = Th$ on $V_z \cap f(R)$. We will say that a function j on $f(R)$ that has this property, i.e. for each z in $f(R)$ one can find a holomorphic extension of j to a full neighbourhood of z in $\mathbb{C}^n$, has the *local extension property*. Suppose conversely that $h \in H^\infty(R)$ is such that Th has that local extension property. The function Th will then be the uniform limit on compact subsets of $V_z \cap f(R)$ of polynomials. However any polynomial on $\mathbb{C}^n$ when restricted to $f(R)$ gives a function of the form Tb for some $b \in B(R)$. Thus at each point of R there is neighbourhood in which h can be locally uniformly approximated by elements of a $B(R)$.That insures that the bounded function h is in the w^*-closure of $B(R)$ in $H^\infty(R)$ which, we just noted, equals $M_{\mathcal{H}_k(R)}$.

We have established the following corollary.

Corollary 16 *Suppose we are in the situation of the previous theorem and (40) holds. If h is a bounded holomorphic function on $f(R)$ and which has the local extension property then there is a bounded function H in H^2_n such that H restricted to $f(R)$ agrees with h; in fact H can be chosen in $M_{H^2_n}$. If the codimension $s = 0$ then the local extension property is automatically satisfied.*

This result applies, for instance, to the maps f used in the proof of Corollary 13. A different type of example is the following. Pick and fix $L > 1$ and let R be the ring domain $R = R_L = \left\{z : L^{-1} < |z| < L\right\}$. Let f be the mapping of R_L into $\mathbb{B}_2$ given by

$$f(z) = c(z, z^{-1}) \text{ with } c = \frac{L^2}{1 + L^4}.$$

In this case $s = 0$. By the theorem $\mathcal{H}_k$ is isomorphic to $H^2(R_L)$ and by the corollary $f(R_L)$ has the extension property.

In fact, for this particular map there is no need for a general theorem. We can define $H^2(R_L)$ using a computationally convenient boundary measure; let $d\sigma_{L^{-1}}$

and $d\sigma_L$ be arc length measure on the two circles which form $\partial\bar{R}$ and set $d\tau = (2\pi L^{-1})^{-1}d\sigma_{L^{-1}} + (2\pi L)^{-1}d\sigma_L$ giving mass 1 to each boundary component. Let $H^2(R_L)$ be the closure in $L^2(\partial\bar{R}, d\tau)$ of the rational functions with poles off R or, equivalently, the closure of the space of polynomials in z and $1/z$. The monomials $\{z^n\}_{n=-\infty}^{\infty}$ are an orthogonal basis for $H^2(A)$ and we have

$$\|z^n\|^2_{H^2(R_L)} = L^{2|n|} + L^{-2|n|}. \tag{41}$$

On the other hand $\mathcal{H}_k(R)$ has reproducing kernels

$$k(z,w) = a_2(f(z), f(w)) = \frac{1}{1 - c\bar{z}w - c\frac{1}{\bar{z}w}}.$$

The norm on $\mathcal{H}_k(R)$ is rotationally invariant and hence the monomials are again an orthogonal basis. Thus to compare $\mathcal{H}_k(R)$ to $H^2(R_L)$ it is enough to compute the norm of the monomials in $\mathcal{H}_k(R)$. Doing a partial fraction decomposition of the reproducing kernel and then a power series expansion gives

$$k(z,w) = \frac{L^4+1}{L^2-1}\sum_{n=-\infty}^{\infty}\frac{(\bar{z}w)^n}{L^{2|n|}}$$

and hence

$$\|z^n\|^2_{\mathcal{H}_k} = \frac{L^2-1}{L^4+1}L^{2|n|}.$$

Comparison with (41) shows that the identity map between the two spaces is an isomorphism.

It is not clear what the natural hypotheses are to insure that (40) holds, however results of B. Lund [26] and E. Stout [35] cover a large category of cases. See also Theorem 3 of E. Bishop in [16].

Theorem 17 *Suppose $B(R)$ contains a nonconstant function h_1 which has modulus identically one on $\partial\bar{R}$. Suppose further that there are $h_2, ..., h_n$ in $B(R)$ so that the mapping $H = (h_1, h_2, ..., h_n)$ separates all but finitely many points of R. Then*

$$\dim(A(R)/B(R)) = s < \infty.$$

If in fact H can be chosen so that H separates every pair of points and the differential dH is nonvanishing then $s = 0$.

Proof. The first statement is in [26], the second in [35]. (The result in [26] is for the case in which H separates all pairs of points. The extension to the more general situation is straightforward.) ■

Other constructions which can be used to form maps f of interest in this context are in [32] and [14].

Remark 18 *It was pointed out to us by John M^{c}Carthy that by using Corollary 13 together with techniques from Chapter 14 of [3] it is possible to prove dilation*

and extension theorems for operators T which have spectrum in $\bar{R}$ and which satisfy the operator inequality

$$I - \sum_{i=1}^{3} f_i(T) f_i(T)^* \geq 0,$$

where the f_i are the functions from the proof of Corollary 13. We plan to return to this issue in a later paper.

3 Inequalities on trees

We now recall some of our earlier results in [9] and [10] on Carleson measures for the Dirichlet space $B_2(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$, as well as for certain $B_2(\mathcal{T})$ spaces on trees $\mathcal{T}$, including the Bergman trees $\mathcal{T}_n$. By a tree we mean a connected loopless graph $\mathcal{T}$ with a root o and a partial order $\leq$ defined by $\alpha \leq \beta$ if α belongs to the geodesic $[o, \beta]$. See for example [9] for more details. We define $B_2(\mathbb{B}_1)$ on the unit disc and $B_2(\mathcal{T})$ on a tree respectively by the norms

$$\|f\|_{B_2(\mathbb{B}_1)} = \left(\int_{\mathbb{B}_1} \left| \left(1 - |z|^2\right) f'(z) \right|^2 \frac{dz}{\left(1 - |z|^2\right)^2} + |f(0)|^2 \right)^{\frac{1}{2}},$$

for f holomorphic on $\mathbb{B}_1$, and

$$\|f\|_{B_2(\mathcal{T})} = \left(\sum_{\alpha \in \mathcal{T}: \alpha \neq o} |f(\alpha) - f(A\alpha)|^2 + |f(o)|^2 \right)^{\frac{1}{2}},$$

for f on the tree $\mathcal{T}$. Here $A\alpha$ denotes the immediate predecessor of α in the tree $\mathcal{T}$. We define the weighted Lebesgue space $L^2_\mu(\mathcal{T})$ on the tree by the norm

$$\|f\|_{L^2_\mu(\mathcal{T})} = \left(\sum_{\alpha \in \mathcal{T}} |f(\alpha)|^2 \mu(\alpha) \right)^{\frac{1}{2}},$$

for f and μ on the tree $\mathcal{T}$. We say that μ is a $B_2(\mathcal{T})$-Carleson measure on the tree $\mathcal{T}$ if $B_2(\mathcal{T})$ embeds continuously into $L^2_\mu(\mathcal{T})$, i.e.

$$\left(\sum_{\alpha \in \mathcal{T}} If(\alpha)^2 \mu(\alpha) \right)^{1/2} \leq C \left(\sum_{\alpha \in \mathcal{T}} f(\alpha)^2 \right)^{1/2}, \quad f \geq 0, \tag{42}$$

or equivalently, by duality,

$$\left(\sum_{\alpha \in \mathcal{T}} I^*(g\mu)(\alpha)^2 \right)^{1/2} \leq C \left(\sum_{\alpha \in \mathcal{T}} g(\alpha)^2 \mu(\alpha) \right)^{1/2}, \quad g \geq 0, \tag{43}$$

where

$$If(\alpha) = \sum_{\beta\in\mathcal{T}:\beta\leq\alpha} f(\beta),$$
$$I^*(g\mu)(\alpha) = \sum_{\beta\in\mathcal{T}:\beta\geq\alpha} g(\beta)\mu(\beta).$$

If (42) is satisfied, we say that μ is a $B_2(\mathcal{T})$-Carleson measure on the tree $\mathcal{T}$. A necessary and sufficient condition for (42) given in [9] is the discrete tree condition

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} I^*\mu(\beta)^2 \leq C^2 I^*\mu(\alpha) < \infty, \qquad \alpha\in\mathcal{T}, \tag{44}$$

which is obtained by testing (43) over $g = \chi_{S_\alpha}$, $\alpha\in\mathcal{T}$. We note that a simpler necessary condition for (42) is

$$d(\alpha) I^*\mu(\alpha) \leq C^2, \tag{45}$$

which one obtains by testing (42) over $f = I^*\delta_\alpha = \chi_{[0,\alpha]}$. However, condition (45) is not in general sufficient for (42) as evidenced by certain Cantor-like measures μ.

We also have the more general two-weight tree theorem from [9].

Theorem 19 *Let w and v be nonnegative weights on a tree $\mathcal{T}$. Then,*

$$\left(\sum_{\alpha\in\mathcal{T}} If(\alpha)^2 w(\alpha)\right)^{1/2} \leq C\left(\sum_{\alpha\in\mathcal{T}} f(\alpha)^2 v(\alpha)\right)^{1/2}, \qquad f\geq 0, \tag{46}$$

if and only if

$$\sum_{\beta\geq\alpha} I^*w(\beta)^2 v(\beta)^{-1} \leq CI^*w(\alpha) < \infty, \qquad \alpha\in\mathcal{T}.$$

We now specialize the tree $\mathcal{T}$ to the Bergman tree $\mathcal{T}_n$ associated with the usual decomposition of the unit ball $\mathbb{B}_n$ into top halves of Carleson "boxes" or Bergman "kubes" K_α. See Subsection 2.2 in [10] and Subsection 2.4 in [36] for details. The following characterization of $B_2(\mathbb{B}_n)$-Carleson measures on the unit ball $\mathbb{B}_n$ is from [10]. Given a positive measure μ on the ball, we denote by $\widehat{\mu}$ the associated measure on the Bergman tree $\mathcal{T}_n$ given by $\widehat{\mu}(\alpha) = \int_{K_\alpha} d\mu$ for $\alpha\in\mathcal{T}_n$. We say that μ is a $B_2(\mathbb{B}_n)$-Carleson measure on the unit ball $\mathbb{B}_n$ if

$$\left(\int_{\mathbb{B}_n} |f(z)|^2 d\mu(z)\right)^{\frac{1}{2}} \leq C\|f\|_{B_2}, \qquad f\in B_2, \tag{47}$$

and that $\widehat{\mu}$ is a $B_2(\mathcal{T}_n)$-Carleson measure on the Bergman tree $\mathcal{T}_n$ if

$$\left(\sum_{\alpha\in\mathcal{T}_n} If(\alpha)^2 \widehat{\mu}(\alpha)\right)^{1/2} \leq C\left(\sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2\right)^{1/2}, \qquad f\geq 0. \tag{48}$$

Theorem 20 *Suppose μ is a positive measure on the unit ball $\mathbb{B}_n$. Then with constants depending only on dimension n, the following conditions are equivalent:*

1. *μ is a $B_2(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$, i.e. (47) holds.*
2. *$\widehat{\mu} = \{\widehat{\mu}(\alpha)\}_{\alpha\in\mathcal{T}_n}$ is a $B_2(\mathcal{T}_n)$-Carleson measure on the Bergman tree $\mathcal{T}_n$, i.e. (48) holds.*
3. *There is $C < \infty$ such that*

$$\sum_{\beta\geq\alpha} I\widehat{\mu}(\beta)^2 \leq CI^*\widehat{\mu}(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n.$$

3.1 Unified proofs for trees

We begin with some notation. Let $\mathcal{G}_{\mathcal{T}}$ be the set of maximal geodesics of $\mathcal{T}$ starting at the root. For $\alpha \in \mathcal{T}$ let $\mathcal{S}(\alpha) \subset \mathcal{G}_{\mathcal{T}}$ denote the collection of all geodesics passing through α (i.e. that are eventually in the successor set $S(\alpha)$). To unify considerations involving both the tree $\mathcal{T}$ and its ideal boundary $\mathcal{G}_{\mathcal{T}}$ we set $\mathcal{T}^* = \mathcal{T}\cup\mathcal{G}_{\mathcal{T}}$ and let $\mathcal{S}^*(\alpha) = S(\alpha)\cup\mathcal{S}(\alpha)$ be the union of the successor set $S(\alpha)$ with its boundary geodesics. We suppose μ, σ, ω, and ν are finite positive measures on $\mathcal{T}^*$ with, for the moment, μ, ω and σ supported in the tree $\mathcal{T}$, and ν supported in the boundary $\mathcal{G}_{\mathcal{T}}$.

We now give a short proof that the two weight tree condition,

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} I^*\mu(\beta)^2\,\omega(\beta) \leq C_0^2 I^*\mu(\alpha) < \infty, \qquad \alpha\in\mathcal{T}, \tag{49}$$

implies the dual Besov-Carleson embedding (which is equivalent to (46) with $\mu = w$ and $\omega = 1/v$).

$$\sum_{\alpha\in\mathcal{T}} I^*(g\mu)(\alpha)^2\,\omega(\alpha) \leq C^2\sum_{\alpha\in\mathcal{T}} g(\alpha)^2\,\mu(\alpha), \qquad g\geq 0, \tag{50}$$

Moreover, we will unify this result and the well-known equivalence of the Hardy-Carleson embedding on the tree,

$$\sum_{\alpha\in\mathcal{T}}\left(\frac{1}{|\mathcal{S}(\alpha)|_\nu}\int_{\mathcal{S}(\alpha)} f d\nu\right)^2 \sigma(\alpha) \leq C^2\int_{\mathcal{G}_{\mathcal{T}}} f^2 d\nu, \qquad f\geq 0 \text{ on } \mathcal{G}_{\mathcal{T}}, \tag{51}$$

with the simple condition on geodesics,

$$\sum_{\beta\geq\alpha}\sigma(\beta) \leq C_0^2\,|\mathcal{S}(\alpha)|_\nu, \qquad \alpha\in\mathcal{T}. \tag{52}$$

We rewrite (50) as

$$\int_{\mathcal{T}}\left(\frac{1}{|\mathcal{S}^*(\alpha)|_\mu}\int_{\mathcal{S}^*(\alpha)} g d\mu\right)^2 |\mathcal{S}^*(\cdot)|_\mu^2\, d\omega(\cdot) \leq C^2\int_{\mathcal{T}^*} g^2 d\mu, \qquad g\geq 0 \text{ on } \mathcal{T}^*, \tag{53}$$

and rewrite (51) as

$$\int_{\mathcal{T}} \left(\frac{1}{|\mathcal{S}^*(\alpha)|_\nu} \int_{\mathcal{S}^*(\alpha)} f d\nu \right)^2 d\sigma(\alpha) \leq C^2 \int_{\mathcal{T}^*} f^2 d\nu, \quad f \geq 0 \text{ on } \mathcal{T}^*. \tag{54}$$

Thus we see that the inequality (53) has exactly the same form as inequality (54), but with $|\mathcal{S}^*(\cdot)|_\mu^2 d\omega(\cdot)$ in place of $d\sigma$ and μ in place of ν. Note that the integrations on the left are over $\mathcal{T}$, where the averages on $\mathcal{S}^*(\alpha)$ are defined. Moreover, the tree condition (49) is just the simple condition (52) for the measures $|\mathcal{S}^*(\cdot)|_\mu^2 d\omega(\cdot)$ and μ:

$$\sum_{\beta \geq \alpha} |\mathcal{S}^*(\beta)|_\mu^2 \omega(\beta) \leq C_0^2 |\mathcal{S}^*(\alpha)|_\mu, \quad \alpha \in \mathcal{T}.$$

In fact, if one permits ν in (54) to live in all of the closure $\mathcal{T}^*$, then we can characterize (54) by a simple condition, and if one permits σ to live in all of $\mathcal{T}^*$ as well, then the corresponding maximal inequality is characterized by a simple condition. The following theorem will be used later to characterize Carleson measures for the Drury-Arveson space $B_2^{1/2}(\mathbb{B}_n)$. The proof can be used to simplify some of the arguments in [9] and [10].

Theorem 21 *Inequality (54) holds if and only if*

$$|S(\alpha)|_\sigma \leq C_0^2 |\mathcal{S}^*(\alpha)|_\nu, \quad \alpha \in \mathcal{T}. \tag{55}$$

More generally, if both σ and ν live in $\mathcal{T}^*$, then the maximal inequality

$$\int_{\mathcal{T}^*} \mathcal{M}f(\zeta)^2 d\sigma(\zeta) \leq C^2 \int_{\mathcal{T}^*} |f|^2 d\nu, \quad \text{for all } f \text{ on } \mathcal{T}^*, \tag{56}$$

where

$$\mathcal{M}f(\zeta) = \mathcal{M}(f d\nu)(\zeta) = \sup_{\alpha \in \mathcal{T}: \alpha \leq \zeta} \frac{1}{|\mathcal{S}^*(\alpha)|_\nu} \int_{\mathcal{S}^*(\alpha)} |f| d\nu,$$

holds if and only if

$$|\mathcal{S}^*(\alpha)|_\sigma \leq C_0^2 |\mathcal{S}^*(\alpha)|_\nu, \quad \alpha \in \mathcal{T}. \tag{57}$$

Proof. The necessity of (55) for (54), and also (57) for (56), follows upon setting $f = \chi_{\mathcal{S}^*(\alpha)}$ in the respective inequality. To see that (57) is sufficient for (56), which includes the assertion that (55) is sufficient for (54), note that the sublinear map $\mathcal{M}$ is bounded with norm 1 from $L^\infty(\mathcal{T}^*; \nu)$ to $L^\infty(\mathcal{T}^*; \sigma)$, and is weak type $1-1$ with constant C_0 by (57). Indeed,

$$\{\zeta \in \mathcal{T}^* : \mathcal{M}f(\zeta) > \lambda\} \subset \cup \{\mathcal{S}^*(\alpha) : \alpha \in \mathcal{T} \text{ and } \mathcal{M}f(\alpha) > \lambda\},$$

and if we let $\lambda > 0$ and denote by Γ the minimal elements in $\{\alpha \in \mathcal{T} : \mathcal{M}f(\alpha) > \lambda\}$, then

$$\begin{aligned} |\{\zeta \in \mathcal{T}^* : \mathcal{M}f(\zeta) > \lambda\}|_\sigma &\leq \sum_{\alpha \in \Gamma} |\mathcal{S}^*(\alpha)|_\sigma \leq C_0^2 \sum_{\alpha \in \Gamma} |\mathcal{S}^*(\alpha)|_\nu \\ &\leq C_0^2 \sum_{\alpha \in \Gamma} \lambda^{-1} \int_{\mathcal{S}^*(\alpha)} |f| d\nu \leq C_0^p \lambda^{-1} \int_{\mathcal{T}^*} |f| d\nu. \end{aligned}$$

Marcinkiewicz interpolation now completes the proof.

The proof actually yields the following more general inequality.

Theorem 22

$$\int_{\mathcal{T}^*} \mathcal{M}f\left(\zeta\right)^2 d\sigma\left(\zeta\right) \leq C^2 \int_{\mathcal{T}^*} \left|f\right|^2 \mathcal{M}\left(d\sigma\right) d\nu, \quad \text{for all } f \text{ on } \mathcal{T}^*.$$

Proof. Following the above proof we use instead the estimate

$$\begin{aligned}
\left|\{\zeta \in \mathcal{T}^* : \mathcal{M}f\left(\zeta\right) > \lambda\}\right|_\sigma &\leq \sum_{\alpha\in\Gamma} \left|\mathcal{S}^*\left(\alpha\right)\right|_\sigma = \sum_{\alpha\in\Gamma} \frac{\left|\mathcal{S}^*\left(\alpha\right)\right|_\sigma}{\left|\mathcal{S}^*\left(\alpha\right)\right|_\nu} \left|\mathcal{S}^*\left(\alpha\right)\right|_\nu \\
&\leq \sum_{\alpha\in\Gamma} \frac{\left|\mathcal{S}^*\left(\alpha\right)\right|_\sigma}{\left|\mathcal{S}^*\left(\alpha\right)\right|_\nu} \lambda^{-1} \int_{\mathcal{S}^*(\alpha)} \left|f\right| d\nu \\
&\leq \sum_{\alpha\in\Gamma} \lambda^{-1} \int_{\mathcal{S}^*(\alpha)} \left|f\left(\zeta\right)\right| \mathcal{M}\left(d\sigma\right)\left(\zeta\right) d\nu\left(\zeta\right),
\end{aligned}$$

which shows that $\mathcal{M}$ is weak type $1-1$ with respect to the measures σ and $\mathcal{M}\left(d\sigma\right) d\nu$.

4 Carleson measures for the Hardy-Sobolev spaces

4.1 The case $\sigma \geq 0$

Given a positive measure μ on the ball, we denote by $\widehat{\mu}$ the associated measure on the Bergman tree $\mathcal{T}_n$ given by $\widehat{\mu}\left(\alpha\right) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}_n$. We will often write $\mu\left(\alpha\right)$ for $\widehat{\mu}\left(\alpha\right)$ when no confusion should arise. Let $\sigma \geq 0$. Recall that μ is a B_2^σ-Carleson measure on $\mathbb{B}_n$ if there is a positive constant C such that

$$\left(\int_{\mathbb{B}_n} \left|f\left(z\right)\right|^2 d\mu\left(z\right)\right)^{\frac{1}{2}} \leq C \left\|f\right\|_{B_2^\sigma}, \tag{58}$$

for all $f \in B_2^\sigma$. In this section we show (Theorem 23) that μ is a B_2^σ-Carleson measure on $\mathbb{B}_n$ if $\widehat{\mu}$ is a $B_2^\sigma(\mathcal{T}_n)$-Carleson measure, i.e. if it satisfies

$$\left(\sum_{\alpha\in\mathcal{T}_n} If\left(\alpha\right)^2 \mu\left(\alpha\right)\right)^{1/2} \leq C \left(\sum_{\alpha\in\mathcal{T}_n} \left[2^{-\sigma d(\alpha)} f\left(\alpha\right)\right]^2\right)^{1/2}, \quad f \geq 0, \tag{59}$$

which is (46) with $w\left(\alpha\right) = \mu\left(\alpha\right)$ and $v\left(\alpha\right) = 2^{-2\sigma d(\alpha)}$. The dual of (59) is

$$\left(\sum_{\alpha\in\mathcal{T}_n} \left[2^{\sigma d(\alpha)} I^* g\mu\left(\alpha\right)\right]^2\right)^{1/2} \leq C \left(\sum_{\alpha\in\mathcal{T}_n} g\left(\alpha\right)^2 \mu\left(\alpha\right)\right)^{1/2}, \quad g \geq 0. \tag{60}$$

Theorem 19 shows that (59) is equivalent to the tree condition

$$\sum_{\beta\geq\alpha} \left[2^{\sigma d(\beta)} I^*\mu\left(\beta\right)\right]^2 \leq CI^*\mu\left(\alpha\right) < \infty, \quad \alpha \in \mathcal{T}_n. \tag{61}$$

Conversely, in the range $0 \leq \sigma < 1/2$, we show that μ is $B_2^\sigma(\mathcal{T}_n)$-Carleson if μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.

Theorem 23 *Suppose $\sigma \geq 0$ and that the structural constants λ, θ in the construction of $\mathcal{T}_n$ (subsection 2.2. of [10]) satisfy $\lambda = 1$ and $\theta = \frac{\ln 2}{2}$. Let μ be a positive measure on the unit ball $\mathbb{B}_n$. Then with constants depending only on σ and n, conditions 2 and 3 below are equivalent, condition 3 is sufficient for condition 1, and provided that $0 \leq \sigma < 1/2$, condition 3 is necessary for condition 1:*

1. *μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$, i.e. (58) holds.*
2. *$\widehat{\mu} = \{\mu(\alpha)\}_{\alpha \in \mathcal{T}_n}$ is a $B_2^\sigma(\mathcal{T}_n)$-Carleson measure, i.e. (59) holds with $\mu(\alpha) = \int_{K_\alpha} d\mu$, where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.*
3. *There is $C < \infty$ such that*

$$\sum_{\beta \geq \alpha} \left[2^{\sigma d(\beta)} I^* \mu(\beta)\right]^2 \leq C I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n,$$

where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.

Proof. The case $\sigma = 0$ is Theorem 20 above, and was proved in [10]. Theorem 19 yields the equivalence of conditions 2 and 3 in Theorem 23.

We use the presentation of $B_2^\sigma(\mathbb{B}_n)$ given by $\mathcal{H}_k$ with kernel function $k(w,z) = \left(\frac{1}{1-\overline{w}\cdot z}\right)^{2\sigma}$ on $\mathbb{B}_n$ as given in Subsubsection 2.3.2. To begin we must verify that this kernel is positive definite, i.e.,

$$\sum_{i,j=1}^{m} a_i \overline{a_j} k(z_i, z_j) \geq 0 \text{ with equality } \Leftrightarrow \text{ all } a_i = 0.$$

Now for $0 < \sigma \leq 1/2$, this follows by expanding $(1 - \overline{w}\cdot z)^{-2\sigma}$ in a power series, using that the coefficients in the expansion are positive, and that the matrices $\left[(\overline{z_i}\cdot z_j)^\ell\right]_{i,j=1}^N$ are nonnegative semidefinite by Schur's theorem for $\ell, N \geq 1$. There is however another approach that not only works for all $\sigma > 0$, but also yields the equivalence of the norms in $\mathcal{H}_k$ and $B_2^\sigma(\mathbb{B}_n)$. For this we recall the invertible "radial" differentiation operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [36] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. If the inverse of $R^{\gamma,t}$ is denoted

$R_{\gamma,t}$, then Proposition 1.14 of [36] yields

$$R^{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}}\right)=\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}}, \tag{62}$$
$$R_{\gamma,t}\left(\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma+t}}\right)=\frac{1}{(1-\overline{w}\cdot z)^{n+1+\gamma}},$$

for all $w\in\mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t. From Theorem 6.1 and Theorem 6.4 of [36] we have that the derivatives $R^{\gamma,m}f(z)$ are "L^2 norm equivalent" to $\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|+f^{(m)}(z)$ for m large enough and $f\in H(\mathbb{B}_n)$. We will also use that the proof of Corollary 6.5 of [36] shows that $R^{\gamma,\frac{n+1+\alpha}{2}-\sigma}$ is a bounded invertible operator from B_2^σ onto the weighted Bergman space A^2_α, provided that neither $n+\gamma$ nor $n+\gamma+\frac{n+1+\alpha}{2}-\sigma$ is a negative integer.

Let $\ell_z^\eta(\zeta)=\left(\frac{1}{1-\overline{z}\cdot\zeta}\right)^\eta$ and set $d\nu_\alpha(\zeta)=(1-|z|^2)^\alpha d\lambda(\zeta)$. Note from (62) that

$$R^{\gamma,t}\ell_z^\eta(\zeta)=\frac{1}{(1-\overline{z}\cdot\zeta)^{n+1+\alpha}}$$

provided $t=n+1+\alpha-\eta$ and $\gamma=\eta-n-1$.The reproducing formula in Theorem 2.7 of [36] yields

$$\begin{aligned}\ell_w^\eta(z)&=\int_{\mathbb{B}_n}\ell_w^\eta(\zeta)\overline{\left(\frac{1}{1-\overline{z}\cdot\zeta}\right)}^{n+1+\alpha}d\nu_\alpha(\zeta)\\&=\int_{\mathbb{B}_n}\ell_w^\eta(\zeta)\,\overline{R^{\gamma,t}\ell_z^\eta(\zeta)}d\nu_\alpha(\zeta).\end{aligned}$$

Now let $S^{\gamma,t}$ be the square root of $R^{\gamma,t}$ defined by

$$S^{\gamma,t}f(z)=\sum_{k=0}^{\infty}\sqrt{\frac{\Gamma(n+1+\gamma)\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\Gamma(n+1+k+\gamma)}}f_k(z).$$

Since $R^{\gamma,t}=(S^{\gamma,t})^*S^{\gamma,t}$ we have with $\eta=2\sigma$ that

$$\begin{aligned}\sum_{i,j=1}^m a_i\overline{a_j}k(x_i,x_j)&=\sum_{i,j=1}^m a_i\overline{a_j}\int_{\mathbb{B}_n}S^{\gamma,t}\ell_{z_i}^\eta(\zeta)\,\overline{S^{\gamma,t}\ell_{z_j}^\eta(\zeta)}d\nu_\alpha(\zeta)\\&=\int_{\mathbb{B}_n}\left|\sum_{i=1}^m a_iS^{\gamma,t}\ell_{z_i}^\eta(\zeta)\right|^2d\nu_\alpha(\zeta)\end{aligned}$$

is positive definite. Note that $S^{\gamma,t}$ is a radial differentiation operator of order $\frac{t}{2}$ so that $S^{\gamma,t}R_{\gamma,\frac{t}{2}}$ is bounded and invertible on the weighted Bergman space A^2_α (e.g. by inspecting coefficients in homogeneous expansions). Thus with $\eta=2\sigma$, we also have the equivalence of norms:

$$\|f\|^2_{\mathcal{H}_k}=\int_{\mathbb{B}_n}\left|S^{\gamma,t}f(\zeta)\right|^2d\nu_\alpha(\zeta)\approx\int_{\mathbb{B}_n}\left|R^{\gamma,\frac{n+1+\alpha-2\sigma}{2}}f(\zeta)\right|^2d\nu_\alpha(\zeta)\approx\|f\|^2_{B_2^\sigma(\mathbb{B}_n)}.$$

For the remainder of this proof we will use the $\mathcal{H}_k$ norm on the space $B_2^\sigma(\mathbb{B}_n)$.

The next part of the argument holds for general Hilbert spaces with reproducing kernel hence we isolate it as a separate lemma. Let $\mathcal{J}$ be a Hilbert space of functions on X with reproducing kernel functions $\{j_x(\cdot)\}_{x\in X}$. A measure μ on X is a $\mathcal{J}$-Carleson measure exactly if the inclusion map T is bounded from $\mathcal{J}$ to $L^2(X,\mu)$.

Lemma 24 *A measure μ is a $\mathcal{J}$-Carleson measure if and only if the linear map*

$$f(\cdot) \to Sf(\cdot) = \int_X \operatorname{Re} j_x(\cdot)\, f(x) d\mu(x)$$

is bounded on $L^2(X,\mu)$.

Proof. T is bounded if and only if the adjoint T^* is bounded from $L^2(X,\mu)$ to $\mathcal{J}$, i.e.

$$\|T^*f\|_{\mathcal{J}}^2 = \langle T^*f, T^*f\rangle_{\mathcal{J}} \le C\|f\|_{L^2(\mu)}^2, \qquad f\in L^2(\mu). \tag{63}$$

For $x\in X$ we have

$$\begin{aligned} T^*f(x) &= \langle T^*f, j_x\rangle_{\mathcal{J}} = \langle f, Tj_x\rangle_{L^2(\mu)} \\ &= \int f(w)\,\overline{j_x(w)} d\mu(w) \\ &= \int j_w(x)\, f(w)\, d\mu(w), \end{aligned}$$

and thus we obtain

$$\begin{aligned} \|T^*f\|_{\mathcal{J}}^2 &= \langle T^*f, T^*f\rangle_{\mathcal{J}} \\ &= \left\langle \int j_w f(w)\, d\mu(w), \int j_{w'} f(w')\, d\mu(w') \right\rangle_{\mathcal{J}} \\ &= \int\int \langle j_w, j_{w'}\rangle_{\mathcal{J}} f(w)\, d\mu(w)\, \overline{f(w')} d\mu(w') \\ &= \int\int j_w(w') f(w)\, d\mu(w)\, \overline{f(w')} d\mu(w'). \end{aligned}$$

Having (63) for general f is equivalent to having it for real f and we now suppose f is real. In that case we continue with

$$\begin{aligned} \|T^*f\|_{\mathcal{J}}^2 &= \int\int \operatorname{Re} j_w(w') f(w) f(w')\, d\mu(w)\, d\mu(w') \\ &= \langle Sf, f\rangle_{L^2(\mu)}. \end{aligned}$$

The last quantity satisfies the required estimates exactly if S is bounded; the proof is complete.

The first of the two following corollaries is immediate from the lemma.

Corollary 25 *Suppose $\mathcal{J}$ and $\mathcal{J}'$ are two reproducing kernel Hilbert spaces on X with kernel functions $\{j\}$ and $\{j'\}$ respectively. If $\operatorname{Re} j_x(y) \leq c \operatorname{Re} j'_x(y)$ then every $\mathcal{J}'$-Carleson measure is a $\mathcal{J}$-Carleson measure. If $\operatorname{Re} j_x(y) \sim c \operatorname{Re} j'_x(y)$ then the two sets of Carleson measures coincide.*

Corollary 26 *Suppose X is a bounded open set in some $\mathbb{R}^k$ and $\partial \bar{X}$ is smooth. Suppose $\mathcal{J}$ and $\mathcal{J}'$ are two reproducing kernel Hilbert spaces on X with kernel functions $\{j\}$ and $\{j'\}$ and that there is a smooth function $h(x,y)$ on $\bar{X} \times \bar{X}$ which is bounded and bounded away from zero so that*

$$j'_x(y) = j_x(y)h(x,y).$$

Then the set of $\mathcal{J}'$-Carleson measures and $\mathcal{J}$-Carleson measures coincide.

Proof. In the proof of the lemma we saw that μ was a $\mathcal{J}$-Carleson measure if and only if

$$f(\cdot) \to Rf(\cdot) = \int_X j_x(\cdot)\ f(x)d\mu(x)$$

was a bounded operator on $L^2(X,\mu)$; and similarly for j'. Thus we need to show that R is bounded if and only if R' is where R' is given by

$$\begin{aligned} f(\cdot) \to R'f(\cdot) &= \int_X j'_x(\cdot)\ f(x)d\mu(x) \\ &= \int_X j_x(\cdot)h(x,\cdot)f(x)d\mu(x). \end{aligned} \tag{64}$$

However this follows from standard facts about bounded operators given by integral kernels. For instance we could extend h to be a smooth compactly supported function in a box in $\mathbb{R}^{2k}$ which contains $X \times X$. Then expand $h(x,y)$ in a multiple Fourier series $\sum_{\alpha=(\alpha_1,\alpha_2)} c_\alpha e^{-i\alpha_1 \cdot x} e^{-i\alpha_2 \cdot y}$ and substitute into (64). This yields the operator equation

$$R' = \sum_{\alpha=(\alpha_1,\alpha_2)} c_\alpha M_{e(\alpha_2)} R M_{e(\alpha_1)}$$

where the c_a are Fourier coefficients and the e's are unimodular characters and the M_e's are the corresponding multiplication operators $M_{e(\alpha)}g(z) = e^{-i\alpha \cdot z} g(z)$. The M_e's are unitary and the smoothness of h insures that $\{c_a\}$ is an absolutely convergent sequence. Hence if R is bounded so is R'. Because h is bounded away from zero we can work with $1/h(x,y)$ to reverse the argument and complete the proof.

In the case of current interest the lemma gives that μ is a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure exactly if we have estimates for

$$\langle T^*f, T^*f \rangle_{B_2^\sigma(\mathbb{B}_n)} = \int\int \operatorname{Re}\left(\frac{1}{1-\overline{w}\cdot w'}\right)^{2\sigma} f(w)\, d\mu(w)\, f(w')\, d\mu(w')$$

for $f \geq 0$.

Now we use that

$$\operatorname{Re}\left(\frac{1}{1-\overline{w}\cdot w'}\right)^{2\sigma} \approx \left|\frac{1}{1-\overline{w}\cdot w'}\right|^{2\sigma} \tag{65}$$

for $0 \leq \sigma < 1/2$, to obtain that μ is $B_2^\sigma(\mathbb{B}_n)$-Carleson if and only if

$$\int\int \left|\frac{1}{1-\overline{w}\cdot w'}\right|^{2\sigma} f(w)\, d\mu(w)\, f(w')\, d\mu(w') \leq C\|f\|^2_{L^2(\mu)}, \qquad f \geq 0.$$

This inequality is easily discretized using that

$$c2^{d(\alpha\wedge\alpha')} \leq \left|\frac{1}{1-\overline{w}\cdot w'}\right| \leq C\int_{\mathcal{U}_n} 2^{d\left(\alpha(Uw)\wedge\alpha\left(Uw'\right)\right)} dU, \tag{66}$$

for $w \in K_\alpha$ and $w' \in K_{\alpha'}$ where $\alpha(Uw)$ denotes the unique kube $K_{\alpha(Uw)}$ containing Uw. The second inequality above is analogous to similar inequalities in Euclidean space used to control an operator by translations of its dyadic version, and the proof is similar (e.g. use (88) below and integrate over $\mathcal{U}_n$). Using this and decomposing the ball $\mathbb{B}_n$ as $\cup_{\alpha\in\mathcal{T}_n} K_\alpha$, we obtain that μ is $B_2^\sigma(\mathbb{B}_n)$-Carleson if and only if

$$\sum_{\alpha,\alpha'\in\mathcal{T}_n} 2^{2\sigma d\left(\alpha\wedge\alpha'\right)} f(\alpha)\,\mu(\alpha)\, f(\alpha')\,\mu(\alpha') \leq C\sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2\,\mu(\alpha), \qquad f \geq 0,$$

where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree. Now for $\sigma > 0$,

$$2^{2\sigma d\left(\alpha\wedge\alpha'\right)} \approx \sum_{\gamma\leq\alpha\wedge\alpha'} 2^{2\sigma d(\gamma)},$$

and so the left side above is approximately

$$\sum_{\alpha,\alpha'\in\mathcal{T}_n}\sum_{\gamma\leq\alpha\wedge\alpha'} 2^{2\sigma d(\gamma)} f(\alpha)\,\mu(\alpha)\, f(\alpha')\,\mu(\alpha') = \sum_{\gamma\in\mathcal{T}_n} 2^{2\sigma d(\gamma)} I^* f(\gamma)^2.$$

Thus for $0 < \sigma < 1/2$, μ is $B_2^\sigma(\mathbb{B}_n)$-Carleson if and only if (60) holds where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree. By Theorem 19, this is equivalent to the tree condition (61) where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree. However, we need only consider a fixed Bergman tree $\mathcal{T}_n$ since if μ is a positive measure on the ball whose discretization $\mu_{\mathcal{T}_n}$ on $\mathcal{T}_n$ satisfies the tree condition, then its discretization $\mu_{U\mathcal{T}_n}$ to any unitary rotation $U\mathcal{T}_n$ also satisfies the tree condition (with a possibly larger, but controlled constant). Indeed, Theorem 19 shows that $\mu_{\mathcal{T}_n}$ is $B_2^\sigma(\mathcal{T}_n)$-Carleson, and hence so is the fattened measure defined by

$$\mu^\natural_{\mathcal{T}_n}(\alpha) = \sum_{d(\alpha,\beta)\leq N} \mu_{\mathcal{T}_n}(\beta), \qquad \alpha \in \mathcal{T}_n.$$

Since $\mu_{U\mathcal{T}_n}$ is pointwise dominated by $\mu^{\natural}_{\mathcal{T}_n}$ for N sufficiently large, $\mu_{U\mathcal{T}_n}$ is $B_2^\sigma(\mathcal{T}_n)$-Carleson as well, hence satisfies the tree condition (61) with $U\mathcal{T}_n$ in place of $\mathcal{T}_n$.

Finally, we note that in the case $\sigma \geq 1/2$, the above argument, together with the inequality

$$\left| \operatorname{Re}\left(\frac{1}{1-\overline{w}\cdot w'} \right)^{2\sigma} \right| \leq \left| \frac{1}{1-\overline{w}\cdot w'} \right|^{2\sigma},$$

shows that the tree condition (61) is sufficient for μ to be a $B_2^\sigma(\mathbb{B}_n)$-Carleson measure. This completes the proof of Theorem 23.

4.2 The case $\sigma = 1/2$: The Drury-Arveson Hardy space H_n^2

The above theorem just misses capturing the Drury-Arveson Hardy space $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$. If we take $\sigma = 1/2$ in the above proof, then (65) combined with the first inequality in (66) is weakened to the inequality

$$\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'} = \frac{\operatorname{Re}(1-\overline{z}\cdot z')}{|1-\overline{z}\cdot z'|^2} \geq c + c2^{2d(\alpha\wedge\alpha')-d^*([\alpha]\wedge[\alpha'])}, \qquad z\in K_\alpha, z'\in K_{\alpha'}, \tag{67}$$

(see below for the definition of $d^*([\alpha]\wedge[\alpha'])$ related to a quotient tree $\mathcal{R}_n$ of the Bergman tree $\mathcal{T}_n$) which does not lead to the tree condition (61). We will however modify the proof so as to give a characterization in Theorem 34 below of the Carleson measures for $H_n^2 = B_2^{1/2}(\mathbb{B}_n)$ in terms of the simple condition (92) and the "split" tree condition (107) given below. We will proceed by three propositions, the first reducing the Carleson measure embedding for H_n^2 to a positive bilinear inequality on the ball.

Proposition 27 *Let μ be a positive measure on the ball $\mathbb{B}_n$. Then μ is H_n^2-Carleson if and only if the bilinear inequality*

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left(\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\right) f(z')\,d\mu(z')\,g(z)\,d\mu(z) \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}, \tag{68}$$

holds for all $f, g \geq 0$. Moreover, provided we use the H_n^2 norm for Carleson measures (but not the $B_2^{1/2}(\mathbb{B}_n)$ norm) the constants implicit in the above statement are independent of dimension n.

Proof. This is immediate from Lemma 24.

We will proceed from the continuous bilinear inequality (68) in two steps. First we obtain Proposition 29 which states that (68) is equivalent to a family of discrete inequalities involving positive quantities. In the section following that we give necessary and sufficient conditions for the discrete inequalities to hold.

However before doing those things we introduce two additional objects associated to the tree $\mathcal{T}_n$. The first is a decomposition of $\mathcal{T}_n$ into a set of equivalence

classes called rings. The rings will help provide a language for a precise description of the local size of the integration kernel in (68). Second, we introduce a notion of a unitary rotation of $\mathcal{T}_n$. As is often the case, when we pass from a discrete inequality to a continuous one technical problems arise associated with edge effects. We will deal with those by averaging over unitary rotations of $\mathcal{T}_n$.

4.2.1 A modified Bergman tree $\mathcal{T}_n$ and its quotient tree $\mathcal{R}_n$

We begin by recalling the main features of the construction of $\mathcal{T}_n$ given in [10], and describe the modification we need. Recall that β is the Bergman metric on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. Note that for each $r > 0$

$$\mathcal{S}_r = \partial B_\beta(0, r) = \{z \in \mathbb{B}_n : \beta(0, z) = r\}$$

is a Euclidean sphere centered at the origin. In fact, by (1.40) in [36] we have $\beta(0, z) = \tanh^{-1}|z|$, and so

$$\begin{aligned} 1 - |z|^2 &= 1 - \tanh^2 \beta(0, z) \\ &= \frac{4}{e^{2\beta(0,z)} + 2 + e^{-2\beta(0,z)}} \\ &\approx 4e^{-2\beta(0,z)} \end{aligned} \tag{69}$$

for $\beta(0, z)$ large. We recall the following elementary abstract construction from [10] (Lemma 7 on page 18).

Lemma 28 *Let (X, d) be a separable metric space and $\lambda > 0$. There is a denumerable set of points $E = \{x_j\}_{j=1}^{\infty \text{ or } J}$ and a corresponding set of Borel subsets Q_j of X satisfying*

$$\begin{aligned} X &= \cup_{j=1}^{\infty \text{ or } J} Q_j, \\ Q_i \cap Q_j &= \phi, \quad i \neq j, \\ B(x_j, \lambda) \subset Q_j &\subset B(x_j, 2\lambda), \quad j \geq 1. \end{aligned} \tag{70}$$

We refer to the sets Q_j as *qubes* centered at x_j. In [10], we applied Lemma 28 to the spheres $\mathcal{S}_r$ for $r > 0$ as follows. Fix *structural constants* $\theta, \lambda > 0$. For $N \in \mathbb{N}$, apply the lemma to the metric space $(\mathcal{S}_{N\theta}, \beta)$ to obtain points $\left\{z_j^N\right\}_{j=1}^J$ and qubes $\left\{Q_j^N\right\}_{j=1}^J$ in $\mathcal{S}_{N\theta}$ satisfying (70). For the remainder of this subsection we assume $\theta = \frac{\ln 2}{2}$ and $\lambda = 1$.

However, we now wish to facilitate the definition of an equivalence relation that identifies qubes "lying in the same complex line intersected with the sphere". To achieve this we recall the projective space $\mathbb{C}P(n-1)$ can be realized as the set of all complex circles $[\zeta] = \left\{e^{is}\zeta : e^{is} \in \mathbb{T}\right\}$, $\zeta \in \partial\mathbb{B}_n$, in the unit sphere (for $n = 2$ these circles give the Hopf fibration of the real 3−sphere). In [5] an induced Koranyi metric was defined on $\mathbb{C}P(n-1)$ by

$$d([\eta], [\zeta]) = \inf\left\{d\left(e^{is}\eta, e^{it}\zeta\right) : e^{is}, e^{it} \in \mathbb{T}\right\}$$

where $d(\eta,\zeta) = |1-\overline{\eta}\cdot\zeta|^{\frac{1}{2}}$. We scale this construction to the sphere $\mathcal{S}_r$ by defining $\mathbb{P}_r$ to be the projective space of complex circles $[\zeta] = \left\{e^{is}\zeta : e^{is}\in\mathbb{T}\right\}$, $\zeta\in\mathcal{S}_r$, in the sphere $\mathcal{S}_r$ with induced Bergman metric

$$\beta([\eta],[\zeta]) = \inf\left\{\beta\left(e^{is}\eta, e^{it}\zeta\right) : e^{is}, e^{it}\in\mathbb{T}\right\}.$$

For $N\in\mathbb{N}$, we now apply Lemma 28 to the projective metric space $(\mathbb{P}_{N\theta},\beta)$ to obtain projective points (complex circles) $\left\{\mathsf{w}_j^N\right\}_{j=1}^J$, J depending on N, in $\mathbb{P}_{N\theta}$ and unit projective qubes $\left\{\mathsf{Q}_j^N\right\}_{j=1}^J$ contained in $\mathbb{P}_{N\theta}$ satisfying (70). For each N and j we define points $\left\{z_{j,i}^N\right\}_{i=1}^M$ on the complex circle w_j^N that are approximately distance 1 from their neighbours in the Bergman metric: $\beta\left(z_{j,i}^N, z_{j,i+1}^N\right)\approx 1$ for $1\leq i\leq M$ ($z_{j,M+1}^N = z_{j,1}^N$). We then define corresponding *qubes* $\left\{Q_{j,i}^N\right\}_i$ so that $\mathsf{Q}_j^N = \cup_i Q_{j,i}^N$, and so that (70) holds in the metric space $(\mathcal{S}_{N\theta},\beta)$ for the collection $\left\{Q_{j,i}^N\right\}_{j,i}$.

For $z\in\mathbb{B}_n$, let $P_r z$ denote the radial projection of z onto the sphere $\mathcal{S}_r$. We now define subsets $K_{j,i}^N$ of $\mathbb{B}_n$ by $K_1^0 = \{z\in\mathbb{B}_n : \beta(0,z)<\theta\}$ and

$$K_{j,i}^N = \left\{z\in\mathbb{B}_n : N\theta\leq d(0,z)<(N+1)\theta,\ P_{N\theta}z\in Q_{j,i}^N\right\},\qquad N\geq 1 \text{ and } j,i\geq 1.$$

We define corresponding points $c_{j,i}^N\in K_{j,i}^N$ by

$$c_{j,i}^N = P_{\left(N+\frac{1}{2}\right)\theta}\left(z_{j,i}^N\right).$$

We will refer to the subset $K_{j,i}^N$ of $\mathbb{B}_n$ as a *kube* centered at $c_{j,i}^N$ (while K_1^0 is centered at 0). Similarly we define *projective kubes* $\mathsf{K}_j^N = \cup_i K_{j,i}^N$ with centre $\mathsf{c}_j^N = P_{\left(N+\frac{1}{2}\right)\theta}\left(\mathsf{w}_j^N\right)$.

Define a tree structure on the collection of all projective kubes

$$\mathcal{R}_n = \left\{\mathsf{K}_j^N\right\}_{N\geq 0, j\geq 1}$$

by declaring that K_i^{N+1} is a child of K_j^N, written $\mathsf{K}_i^{N+1}\in\mathcal{C}\left(\mathsf{K}_j^N\right)$, if the projection $P_{N\theta}\left(\mathsf{w}_i^{N+1}\right)$ of the circle w_i^{N+1} onto the sphere $\mathcal{S}_{N\theta}$ lies in the projective qube Q_j^N. In the case $N=0$, we declare every kube K_j^1 to be a child of the root kube K_1^0. An element K_j^N is, roughly, the orbit of a single kube under a circle action; thus we often refer to them as rings and to $\mathcal{R}_n$ as the ring tree. One can think of the ring tree $\mathcal{R}_n$ as a "quotient tree" of the Bergman tree $\mathcal{T}_n$ by the one-parameter family of slice rotations $z\to e^{is}z$, $e^{is}\in\mathbb{T}$.

We will now define a tree structure on the collection of kubes

$$\mathcal{T}_n = \left\{K_{j,i}^N\right\}_{N\geq 0 \text{ and } j,i\geq 1}$$

that is compatible with the above tree structure on the collection of projective kubes $\mathcal{R}_n$. To this end, we reindex the kubes $\left\{K_{j,i}^N\right\}_{N\geq 0 \text{ and } j,i\geq 1}$ as $\left\{K_j^N\right\}_{N\geq 0, j\geq 1}$ and define an equivalence relation $\sim$ on the reindexed collection $\left\{K_j^N\right\}_j$ by

declaring kubes equivalent that lie in the same projective kube: $K_i^N \sim K_k^N$ if and only if there is a projective kube K_j^N such that $K_i^N, K_k^N \in \mathsf{K}_j^N$. Given $K_i^N \in \mathcal{T}_n$, we denote by $\left[K_i^N\right]$ the equivalence class of K_i^N, which can of course be identified with a projective kube in $\mathcal{R}_n$. Define the tree structure on $\mathcal{T}_n$ by declaring that K_i^{N+1} is a child of K_j^N, written $K_i^{N+1} \in \mathcal{C}\left(K_j^N\right)$, if the projection $P_{N\theta}\left(z_i^{N+1}\right)$ of z_i^{N+1} onto the sphere $\mathcal{S}_{N\theta}$ lies in the qube Q_j^N. Note that by construction, it follows that $\left[K_i^{N+1}\right]$ is then also a child of $\left[K_j^N\right]$ in $\mathcal{R}_n$. In the case $N = 0$, we declare every kube K_j^1 to be a child of the root kube K_1^0.

We will typically write α, β, γ etc. to denote elements K_j^N of the tree $\mathcal{T}_n$ when the correspondence with the unit ball $\mathbb{B}_n$ is immaterial. We will write K_α for the kube K_j^N and c_α for its center c_j^N when the correspondence matters. Sometimes we will further abuse notation by using α to denote the center $c_\alpha = c_j^N$ of the kube $K_\alpha = K_j^N$. Similarly, we will typically write A, B, C etc. to denote elements K_j^N of the ring tree $\mathcal{R}_n$ when the correspondence with the unit ball $\mathbb{B}_n$ is immaterial, and we will write K_A for the projective kube K_j^N corresponding to A when the correspondence matters. Finally, for $\alpha \in \mathcal{T}_n$, we denote by $[\alpha]$ the ring in $\mathcal{R}_n$ that corresponds to the equivalence class of α. The following compatibility relations hold for $\alpha, \beta \in \mathcal{T}_n$ and $A, B \in \mathcal{R}_n$:

$$\begin{aligned} &\beta \leq \alpha \Longrightarrow [\beta] \leq [\alpha], \\ &B \leq A \Longleftrightarrow \text{ for every } \alpha \in A \text{ there is } \beta \in B \text{ with } \beta \leq \alpha, \end{aligned} \tag{71}$$

We will also need the notion of a unitary rotation of $\mathcal{T}_n$. For each $w \in \mathbb{B}_n$ define $\langle w \rangle \in \mathcal{T}_n$ to be the unique tree element such that $w \in K_{\langle w \rangle}$, and define $[w] \in \mathcal{R}_n$ to be the unique ring tree element such that $w \in \mathsf{K}_{[w]}$ (here we are viewing the projective kube $\mathsf{K}_{[w]}$ as a subset of the ball $\mathbb{B}_n$). The notation is coherent; the ring containing w is the equivalence class in $\mathcal{T}$ containing the kube $K_{\langle w \rangle}$; $[w] = [\langle w \rangle]$. Let $\mathcal{U}_n$ be the unitary group with Haar measure dU. Recall that we may identify α with the center c_α of the Bergman kube K_α (subsubsection 5.2.1 of [10]). If we define $K_{U^{-1}\alpha} = U^{-1}K_\alpha$, then $\{K_{U^{-1}\alpha}\}_{\alpha \in \mathcal{T}_n} \equiv \left\{U^{-1}K_\alpha\right\}_{\alpha \in \mathcal{T}_n}$ is the Bergman grid rotated by U^{-1}, and

$$\alpha = \langle Uz \rangle \Leftrightarrow Uz \in K_\alpha \Leftrightarrow z \in U^{-1}K_\alpha \Leftrightarrow z \in K_{U^{-1}\alpha}. \tag{72}$$

We denote by $U^{-1}\mathcal{T}_n$ the tree corresponding to the rotated grid $\{K_{U^{-1}\alpha}\}_{\alpha \in \mathcal{T}_n}$. The same construction applies to obtain the rotated ring tree $U^{-1}\mathcal{R}_n$, and the compatibility relation (71) persists between $U^{-1}\mathcal{T}_n$ and $U^{-1}\mathcal{R}_n$ since $\left[U^{-1}\alpha\right] = U^{-1}[\alpha]$. We also define $\langle w \rangle_U \in U^{-1}\mathcal{T}_n$ and $[w]_U \in U^{-1}\mathcal{R}_n$ by $w \in K_{\langle w \rangle_U}$ and $w \in \mathsf{K}_{[w]_U}$ respectively. Then from (72) we have $\alpha = \langle Uz \rangle \Leftrightarrow U^{-1}\alpha = \langle z \rangle_U$.

We will also want distance functions with controlled behavior under unitary rotations. We now extend the definition of the tree distance $d_{U^{-1}\mathcal{T}_n}$ and the ring distance $d_{U^{-1}\mathcal{R}_n}$ on the rotations $U^{-1}\mathcal{T}_n$ and $U^{-1}\mathcal{R}_n$ to $\mathbb{B}_n \times \mathbb{B}_n$ by

$$\begin{aligned} d_{U^{-1}\mathcal{T}_n}(z, w) &= d_{U^{-1}\mathcal{T}_n}\left(\langle z \rangle_U, \langle w \rangle_U\right), \qquad z, w \in \mathbb{B}_n, \\ d_{U^{-1}\mathcal{R}_n}(z, w) &= d_{U^{-1}\mathcal{R}_n}\left([z]_U, [w]_U\right) \qquad z, w \in \mathbb{B}_n. \end{aligned}$$

We have the following identities:

$$d_{U^{-1}\mathcal{T}_n}(z,w) = d_{\mathcal{T}_n}(Uz,Uw),$$
$$d_{U^{-1}\mathcal{R}_n}(z,w) = d_{\mathcal{R}_n}(Uz,Uw).$$

We often write simply d when the underlying tree is evident, especially when it is $\mathcal{T}_n$ or $\mathcal{R}_n$, and provided this will cause no confusion; e.g. $d(z,w) = d_{\mathcal{T}_n}(\langle z\rangle, \langle w\rangle)$.

Finally, we introduce yet another structure on the trees $\mathcal{T}_n$ and $\mathcal{R}_n$, namely the *unitary* tree distance d^* given by

$$d^*(\alpha,\beta) = \inf_{U\in\mathcal{U}_n} d_{\mathcal{T}_n}(Uc_\alpha, Uc_\beta) = \inf_{U\in\mathcal{U}_n} d_{U^{-1}\mathcal{T}_n}(c_\alpha, c_\beta),$$
$$d^*([\alpha],[\beta]) = \inf_{U\in\mathcal{U}_n} d_{\mathcal{R}_n}(Uc_\alpha, Uc_\beta) = \inf_{U\in\mathcal{U}_n} d_{U^{-1}\mathcal{R}_n}(c_\alpha, c_\beta).$$

Note that the analogous definitions of d^* on the rotated trees $U^{-1}\mathcal{T}_n$ and $U^{-1}\mathcal{R}_n$ coincide with the above definitions, so that we can write simply d^* for $d^*_{U^{-1}\mathcal{T}_n}$ or $d^*_{U^{-1}\mathcal{R}_n}$ without ambiguity. We now **define** $d^*(\alpha\wedge\beta)$ and $d^*(A\wedge B)$ in analogy with the corresponding formulas for d; namely

$$2d^*(\alpha\wedge\beta) = d^*(\alpha) + d^*(\beta) - d^*(\alpha,\beta), \qquad \alpha,\beta\in U^{-1}\mathcal{T}_n,$$
$$2d^*(A\wedge B) = d^*(A) + d^*(B) - d^*(A,B), \qquad A,B\in U^{-1}\mathcal{R}_n,$$

so that

$$d^*(\alpha\wedge\beta) = \sup_{U\in\mathcal{U}_n} d_{U^{-1}\mathcal{T}_n}\left(\langle c_\alpha\rangle_U \wedge \langle c_\beta\rangle_U\right), \qquad \alpha,\beta\in\mathcal{T}_n,$$
$$d^*([\alpha]\wedge[\beta]) = \sup_{U\in\mathcal{U}_n} d_{U^{-1}\mathcal{R}_n}\left([c_\alpha]_U \wedge [c_\beta]_U\right) \qquad \alpha,\beta\in\mathcal{T}_n,$$

The unitary distance d^* on the ring tree $\mathcal{R}_n$ will play a crucial role in discretizing the bilinear inequality (68) in the next section. (Actually $d^*([\alpha]\wedge[\beta])$ is a function of the pair $([\alpha],[\beta])$ not of the ring tree element $[\alpha]\wedge[\beta]$. We indulge in this slight abuse of notation because below $d^*([\alpha]\wedge[\beta])$ will have the role of a substitute for $d([\alpha]\wedge[\beta])$.)

4.2.2 The discrete inequality

We can now state the discretization inequality.

Proposition 29 *Let μ be a positive measure on $\mathbb{B}_n$. Then the bilinear inequality (68) is equivalent to having, for all unitary rotations of a fixed Bergman tree $\mathcal{T}_n$ together with the corresponding rotations of the associated ring tree $\mathcal{R}_n$, and with constants independent of the rotation, the discrete inequality,*

$$\sum_{\alpha\in\mathcal{T}_n} |T_\mu g(\alpha)|^2 \mu(\alpha) \le C \sum_{\alpha\in\mathcal{T}_n} |g(\alpha)|^2 \mu(\alpha), \qquad g\ge 0, \tag{73}$$

where T_μ is the positive linear operator on the tree $\mathcal{T}_n$ given by,

$$T_\mu g(\alpha) = \sum_{\beta \in \mathcal{T}_n} 2^{2d(\alpha\wedge\beta)-d^*([\alpha]\wedge[\beta])} g(\beta)\,\mu(\beta), \qquad \alpha \in \mathcal{T}_n. \tag{74}$$

Equivalently, (73) can be replaced by the bilinear estimate

$$\begin{aligned} &\sum_{\alpha,\alpha' \in \mathcal{T}_n} 2^{2d(\alpha\wedge\alpha')-d^*([\alpha]\wedge[\alpha'])} f(\alpha)\,\mu(\alpha)\, g(\alpha')\,\mu(\alpha') \qquad (75)\\ &\leq C \left\{ \sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2\,\mu(\alpha) \right\}^{\frac{1}{2}} \left\{ \sum_{\alpha'\in\mathcal{T}_n} g(\alpha')^2\,\mu(\alpha') \right\}^{\frac{1}{2}}, \end{aligned}$$

where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.

Proof. We first establish (75), i.e. we discretize the bilinear inequality (68) to the following discrete bilinear inequality valid for all unitary rotations $U^{-1}\mathcal{T}_n$ of the Bergman tree $\mathcal{T}_n$:

$$\begin{aligned} &\sum_{\alpha,\alpha' \in U^{-1}\mathcal{T}_n} 2^{2d(\alpha\wedge\alpha')-d^*([\alpha]\wedge[\alpha'])} f(\alpha)\,\mu(\alpha)\, g(\alpha')\,\mu(\alpha') \qquad (76)\\ &\leq C \left\{ \sum_{\alpha\in U^{-1}\mathcal{T}_n} f(\alpha)^2\,\mu(\alpha) \right\}^{\frac{1}{2}} \left\{ \sum_{\alpha'\in U^{-1}\mathcal{T}_n} g(\alpha')^2\,\mu(\alpha') \right\}^{\frac{1}{2}}, \end{aligned}$$

for all $U \in \mathcal{U}_n$, $f, g \geq 0$ on $U^{-1}\mathcal{T}_n$ and where the constant C is independent of U, f, g. At a crucial point in the argument below, we need to estimate the distance $1 - |\overline{z}\cdot z'|^2$ in terms of the tree structure, and this is what leads to the associated ring tree $\mathcal{R}_n$ and the quantities $d([\alpha]\wedge[\alpha'])$ and $d^*([\alpha]\wedge[\alpha'])$. Recall that a *slice* of the ball $\mathbb{B}_n$ is the intersection of the ball with a complex line through the origin. In particular, every point $z \in \mathbb{B}_n \setminus \{0\}$ lies in a unique slice

$$S_z = \left\{ \left(e^{i\theta} z_1, ..., e^{i\theta} z_n\right) : \theta \in [0, 2\pi) \right\}.$$

We define two elements α and α' of the Bergman tree $\mathcal{T}_n$ to be *slice-related* if $\alpha \sim \alpha'$ where, recall, $\sim$ denotes that the two elements lie in the same projective kube. Now given $\alpha, \alpha' \in \mathcal{T}_n$, let

$$[o,\alpha] = \{o, \alpha_1, ..., \alpha_m = \alpha\} \text{ and } [o,\alpha'] = \{o, \alpha'_1, ..., \alpha'_{m'} = \alpha'\}$$

be the geodesics from the root o to α, α' respectively. We then have from (71) that α_k and α'_k are slice-related if and only if $k \leq d([\alpha]\wedge[\alpha'])$.

It may help the reader to visualize $d([\alpha]\wedge[\alpha'])$ in the following way. Imagine that each slice S is thickened to a *slab* $\mathcal{S}$ of width one in the Bergman metric. Thus in the Euclidean metric, a slab $\mathcal{S}$ is a lens whose "thickness" at any point is roughly the square root of the distance to the boundary of the ball $\partial\mathbb{B}_n$. Moreover, given $z \in \mathbb{B}_n$, we denote by $\mathcal{S}_z$ the slab corresponding to the

slice S_z, but truncated by intersecting it with $B(0,|z|)$. The slabs $\mathcal{S}_{c_\alpha}$ and $\mathcal{S}_{c_{\alpha'}}$ associated with the unique slices S_{c_α} and $S_{c_{\alpha'}}$ through c_α and $c_{\alpha'}$ will intersect in a "disc" of radius roughly $d([\alpha]\wedge[\alpha'])$ in the Bergman metric - at least this will be the case for a "fixed proportion" of pairs (α,α'), and will be literally true for all pairs with the unitary quantity $d^*([\alpha]\wedge[\alpha'])$ in place of $d([\alpha]\wedge[\alpha'])$. Note from this picture that $\alpha_{d([\alpha]\wedge[\alpha'])}$ is the *exit point* $E_{\alpha'}\alpha$ of the geodesic $[o,\alpha]$ from the slab $\mathcal{S}_{\alpha'}$ associated to the slice $S_{\alpha'}$ through $c_{\alpha'}$, and similarly, $\alpha'_{d([\alpha]\wedge[\alpha'])}$ is the exit point $E_\alpha\alpha'$ of the geodesic $[o,\alpha']$ from the slab $\mathcal{S}_\alpha$. Both points have the same distance from the root. Note that we can also define $E_{\alpha'}\alpha$ as the intersection of the geodesic $[o,\alpha]$ with the ring $[\alpha]\wedge[\alpha']$, which we will denote by $E_{[\alpha]\wedge[\alpha']}\alpha$. Finally, note that since $d([\alpha]\wedge[\alpha']) = d(E_{\alpha'}\alpha) = d(E_\alpha\alpha')$ and $\alpha\wedge\alpha' = \alpha_\ell$ where $\ell = \max\{k : \alpha_k = \alpha'_k\}$, we have that $d([\alpha]\wedge[\alpha'])$ satisfies

$$d(\alpha\wedge\alpha') \leq d([\alpha]\wedge[\alpha']) \leq \min\{d(\alpha), d(\alpha')\}. \tag{77}$$

The key feature of the quantity $d([\alpha]\wedge[\alpha'])$ is that $2^{-d\left([\alpha]\wedge\left[\alpha'\right]\right)}$ is essentially $1-|\overline{z}\cdot z'|^2$ for $z\in K_\alpha$, $z'\in K_{\alpha'}$. More precisely, for each $z,z'\in\mathbb{B}_n$, there is a subset Σ of the unitary group $\mathcal{U}_n$ with Haar measure $|\Sigma|\geq c>0$ and satisfying

$$\begin{aligned} c2^{-d\left([Uz]\wedge\left[Uz'\right]\right)} &\leq 1-|\overline{z}\cdot z'|^2, \quad U\in\Sigma, \\ 1-|\overline{z}\cdot z'|^2 &\leq C2^{-d\left([Uz]\wedge\left[Uz'\right]\right)}, \quad U\in\mathcal{U}_n. \end{aligned} \tag{78}$$

In particular, in terms of the unitary ring distance d^*, we have the equivalence

$$1-|\overline{z}\cdot z'|^2 \approx 2^{-d^*\left([z]\wedge\left[z'\right]\right)}. \tag{79}$$

The full force of the first inequality in (78) will not be used until the next subsubsection when we prove the sufficiency of the simple condition and split tree condition for (75). To prove (78), let $S=S_z$ be the slice through z, $\mathcal{S}=\mathcal{S}_z$ the corresponding slab, denote by P projection from the ball onto S, and by Q its orthogonal projection, so that

$$Pw = \frac{\overline{z}\cdot w}{|z|^2}z, \qquad Qw = w-Pw.$$

If $d=d([z]\wedge[z'])$, then $\langle z'\rangle_d$ is the exit point $E_{\langle z\rangle}\langle z'\rangle$ of $[o,\langle z'\rangle]$ from the slab $\mathcal{S}$. Since $\mathcal{S}$ is a lens whose Euclidean "thickness" at any point is roughly the square root of the distance from the boundary, we have

$$\left|Q\left(c_{\langle z'\rangle_d}\right)\right| \leq C2^{-\frac{1}{2}d\left(\langle z'\rangle_d\right)} = C2^{-\frac{1}{2}d}.$$

Since $z'\in K_{\langle z'\rangle}$ where $\langle z'\rangle \geq \langle z'\rangle_{d+1}$, we also have

$$|Q(z')| \leq C2^{-\frac{1}{2}d}.$$

It now follows that

$$\begin{aligned} 1-|\overline{z}\cdot z'|^2 &= 1-|z|^2|Pz'|^2 = 1-|z|^2\left(1-|Qz'|^2\right) \\ &\leq 1-|z|^2+C^2 2^{-d} \\ &\leq C^2 2^{-d\left([z]\wedge\left[z'\right]\right)}, \end{aligned}$$

and since this argument works for any Bergman tree $U^{-1}\mathcal{T}_n$, this yields the second inequality in (78).

To obtain the first inequality in (78), we use a standard averaging argument as follows. Given $U \in \mathcal{U}_n$, if $d = d([Uz] \wedge [Uz'])$, then $(\langle z' \rangle_U)_d$ is the exit point $E_{\langle z \rangle_U} \langle z' \rangle_U$ of $[o, \langle z' \rangle_U]$ from the slab $\mathcal{S}$. Since $c_{\left(\langle z' \rangle_U\right)_{d+1}}$ lies outside $\mathcal{S}$, and since $\mathcal{S}$ is a lens whose Euclidean "thickness" at any point is roughly the square root of the distance from the boundary, we have

$$\left| Q\left(c_{\left(\langle z' \rangle_U\right)_{d+1}} \right) \right| \geq c 2^{-\frac{1}{2} d\left(\left(\langle z' \rangle_U \right)_d \right)} = c 2^{-\frac{1}{2} d}.$$

Since $z' \in K_{\langle z' \rangle_U}$ where $\langle z' \rangle_U \geq (\langle z' \rangle_U)_{d+1}$, we thus also have

$$|Q(z')| \geq c 2^{-\frac{1}{2} d},$$

for U in a subset Σ of the unitary group $\mathcal{U}_n$ such that $|\Sigma| \geq c > 0$ (the third line of (70) is used here). It now follows that

$$\begin{aligned} 1 - |\overline{z} \cdot z'|^2 &= 1 - |z|^2 |Pz'|^2 = 1 - |z|^2 \left(1 - |Qz'|^2\right) \\ &\geq 1 - |z|^2 \left(1 - c^2 2^{-d}\right) \\ &\geq c^2 2^{-d} = c^2 2^{-d\left([Uz] \wedge \left[Uz'\right]\right)}, \end{aligned}$$

for all $U \in \Sigma$, which yields the first inequality in (78).

The main inequalities used in establishing the equivalence of (68) and (76) are (67), i.e.

$$\operatorname{Re} \frac{1}{1 - \overline{z} \cdot z'} \geq c + c 2^{2d\left(\alpha \wedge \alpha'\right) - d^*\left([\alpha] \wedge \left[\alpha'\right]\right)}, \qquad z \in K_\alpha, z' \in K_{\alpha'}, \tag{80}$$

for all $\alpha, \alpha' \in U^{-1}\mathcal{T}_n$, $U \in \mathcal{U}_n$, together with a converse obtained by averaging over all unitary rotations $U^{-1}\mathcal{T}_n$ of the Bergman tree $\mathcal{T}_n$,

$$\operatorname{Re} \frac{1}{1 - \overline{z} \cdot z'} \leq C + C \int_{\mathcal{U}_n} 2^{2d\left(\langle Uz \rangle \wedge \left\langle Uz' \right\rangle\right) - d^*\left([z] \wedge \left[z'\right]\right)} dU. \tag{81}$$

This latter inequality is analogous to similar inequalities in Euclidean space used to control an operator by translations of its dyadic version, and the proof given below is similar.

To prove (80) and (81), we will use the identity (Lemma 1.3 of [36])

$$1 - \overline{\varphi_a(w)} \cdot \varphi_a(z) = \frac{(1 - \overline{a} \cdot a)(1 - \overline{w} \cdot z)}{(1 - \overline{w} \cdot a)(1 - \overline{a} \cdot z)}, \qquad z, w \in \overline{\mathbb{B}_n}, a \in \mathbb{B}_n, \tag{82}$$

the fact that the Bergman balls $B_\beta(a, r)$ are the ellipsoids ([31], page 29)

$$B_\beta(a, r) = \left\{ z \in \mathbb{B}_n : \frac{|P_a z - c_a|^2}{t^2 \rho_a^2} + \frac{|Q_a z|^2}{t^2 \rho_a} < 1 \right\}, \tag{83}$$

where

$$c_a = \frac{\left(1-t^2\right)a}{1-t^2\left|a\right|^2},\ \rho_a = \frac{1-\left|a\right|^2}{1-t^2\left|a\right|^2},$$

and $t > 0$ satisfies $B_\beta(0,r) = B(0,t)$, and the fact that the projection of $B_\beta(a,1)$ onto the sphere $\partial\mathbb{B}_n$ is essentially the nonisotropic Koranyi ball $Q\left(\frac{a}{|a|}, \sqrt{1-|a|^2}\right)$ given in (4.11) of [36] by

$$Q(\zeta,\delta) = \left\{\eta \in \partial\mathbb{B}_n : \left|1-\overline{\eta}\cdot\zeta\right|^{\frac{1}{2}} \leq \delta\right\}, \qquad \zeta \in \partial\mathbb{B}_n. \tag{84}$$

Indeed, if c_α is the center of the Bergman kube K_α, then the successor set $S(\alpha) = \cup_{\beta\geq\alpha}K_\beta$ consists essentially of all points z lying between K_α and its projection onto the sphere, and from (83) and (84) we then have

$$S(\alpha) \approx \left\{z \in \mathbb{B}_n : \left|1-\overline{c_\alpha}\cdot z\right| \leq 1-\left|c_\alpha\right|^2 \approx 2^{-d(\alpha)}\right\}$$

in the sense that if

$$\mathcal{S}_C(w) = \left\{z \in \mathbb{B}_n : \left|1-\overline{w}\cdot z\right| \leq C\left(1-\left|w\right|^2\right)\right\},$$

then there are positive constants c and C such that

$$\mathcal{S}_c(c_\alpha) \subset S(\alpha) \subset \mathcal{S}_C(c_\alpha),$$

where

$$\mathcal{S}_C(c_\alpha) \approx \left\{z \in \mathbb{B}_n : \left|1-\overline{c_\alpha}\cdot z\right| \leq C2^{-d(\alpha)}\right\}.$$

Using (82) with $a = c_\alpha$, $\omega = \varphi_a(w)$ and $\zeta = \varphi_a(z)$, we see that

$$\left|1-\overline{\omega}\cdot\zeta\right| \leq C, \qquad \omega,\zeta \in K_\alpha,$$

since $|w|, |z| \leq \rho < 1$ for $\omega,\zeta \in K_\alpha$, and it now follows easily that

$$\begin{aligned} &\left|1-\overline{\omega}\cdot\zeta\right| \leq C2^{-d(\alpha)}, \qquad \omega,\zeta \in S(\alpha), \\ &\left|1-\overline{\omega}\cdot\zeta\right| \geq c2^{-d(\alpha)}, \qquad \omega \in S(\alpha), \zeta \notin \mathcal{S}_{2C}(\alpha). \end{aligned} \tag{85}$$

Now fix $U \in \mathcal{U}_n$, $\alpha \in U^{-1}\mathcal{T}_n$, $\alpha' \in U^{-1}\mathcal{T}_n$, $z \in K_\alpha$, $z' \in K_{\alpha'}$ and let $\beta = \alpha\wedge\alpha'$ be the minimum of α and α' in the Bergman tree. From the first inequality in (85), we obtain

$$\left|1-\overline{z'}\cdot z\right| \leq C2^{-d(\beta)} = C2^{-d\left(\alpha\wedge\alpha'\right)}. \tag{86}$$

We now write $\frac{\overline{z}\cdot z'}{|\overline{z}\cdot z'|} = e^{i\theta}$, where by localizing z and z' to lie close together near the boundary of the ball, we may assume that both $|\theta|$ and $1-\left|\overline{z'}\cdot z\right|^2$ are small,

say less than $\varepsilon > 0$. We then have

$$\begin{aligned}
\operatorname{Re}\left(1 - \overline{z} \cdot z'\right) &= \left(1 - |\overline{z} \cdot z'|\right) + |\overline{z} \cdot z'| \left(1 - \cos\theta\right) \\
&\approx \left(1 - |\overline{z} \cdot z'|^2\right) + \left(1 - \cos^2\theta\right) \\
&= \left(1 - |\overline{z} \cdot z'|^2\right) + \sin^2\theta \\
&\approx \left(1 - |\overline{z} \cdot z'|^2\right) + |\operatorname{Im}\left(1 - \overline{z} \cdot z'\right)|^2 \\
&= 1 - |\overline{z} \cdot z'|^2 + |1 - \overline{z} \cdot z'|^2 - |\operatorname{Re}\left(1 - \overline{z} \cdot z'\right)|^2 .
\end{aligned}$$

However, for $\varepsilon > 0$ sufficiently small, we may absorb the last term $|\operatorname{Re}\left(1 - \overline{z} \cdot z'\right)|^2$ on the right side into the left side, to obtain

$$\operatorname{Re} \frac{1}{1 - \overline{z} \cdot z'} = \frac{\operatorname{Re}\left(1 - \overline{z} \cdot z'\right)}{|1 - \overline{z} \cdot z'|^2} \approx \frac{1 - |\overline{z} \cdot z'|^2}{|1 - \overline{z} \cdot z'|^2} + 1. \tag{87}$$

Note that (87) persists for all $z, z' \in \mathbb{B}_n$ since if z and z' do not lie close together near the boundary of the ball, then $|1 - \overline{z} \cdot z'| \geq c > 0$.

Using (86), (87) and (79), we immediately have the lower bound

$$\operatorname{Re} \frac{1}{1 - \overline{z} \cdot z'} \geq c + c 2^{2d\left(\alpha \wedge \alpha'\right) - d^*\left([\alpha] \wedge [\alpha']\right)}, \qquad z \in K_\alpha, z' \in K_{\alpha'},$$

which is (80). To obtain the converse (81), we use the third line in (70) to note that for fixed $z, z' \in \mathbb{B}_n$, there is a subset Σ of the unitary group $\mathcal{U}_n$ having Haar measure bounded below by a positive constant c, and such that for each $U \in \Sigma$, if $\alpha \in U^{-1}\mathcal{T}_n$, $\alpha' \in U^{-1}\mathcal{T}_n$, $z \in K_\alpha$, $z' \in K_{\alpha'}$, and $\beta = \alpha \wedge \alpha' \in U^{-1}\mathcal{T}_n$, then z and z' do not lie in a common child $\gamma \in U^{-1}\mathcal{T}_n$ of β (we may of course replace "child" by an "ℓ-fold grandchild" with ℓ sufficiently large and fixed). From the second inequality in (85), we then obtain

$$\left|1 - \overline{z'} \cdot z\right| \geq c 2^{-d\left(\alpha \wedge \alpha'\right)}, \qquad U \in \Sigma, \tag{88}$$

and combined with the second inequality in (78), (87) now yields (81) upon integrating over Haar measure and using $|\Sigma| \geq c > 0$.

Now (68) is invariant under unitary transformations, and so (80) for the tree $U^{-1}\mathcal{T}_n$ immediately shows that (68) implies (76) (note that we are throwing away the constant lower bound of c in (80)).

Conversely, for $U \in \mathcal{U}_n$ let $f\left(U^{-1}\alpha\right) = \int_{U^{-1}K_\alpha} f d\lambda_n$ and $\nu\left(U^{-1}\alpha\right) = \int_{U^{-1}K_\alpha} d\nu$ be the function and measure discretizations of f and ν respectively on the rotated Bergman grid $\{K_{U^{-1}\alpha}\}_{\alpha \in \mathcal{T}_n}$. From (81) and (72) the left side of

(68) with $f = g$ satisfies

$$\begin{aligned}
&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left(\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\right) f(z')\,d\mu(z')\,f(z)\,d\mu(z)\\
&\quad\leq C\int_{\mathcal{U}_n}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n} f(z')\,d\mu(z')\,f(z)\,d\mu(z)\,dU\\
&\qquad + C\int_{\mathcal{U}_n}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{2^{2d\left(\langle Uz\rangle\wedge\left\langle Uz'\right\rangle\right)}}{2^{d^*([z]\wedge[z'])}} f(z')\,d\mu(z')\,f(z)\,d\mu(z)\,dU\\
&\quad= I + II.
\end{aligned}$$

Now μ is a finite measure and from Cauchy's inequality, we obtain that

$$I \leq C\left\|f\right\|_{L^2(\mu)}^2. \tag{89}$$

For each $U \in \mathcal{U}_n$, we decompose the ball $\mathbb{B}_n$ by the rotated Bergman tree $U^{-1}\mathcal{T}_n$ to obtain

$$II = C\int_{\mathcal{U}_n}\sum_{\alpha,\alpha'\in\mathcal{T}_n}\int_{z\in K_{U^{-1}\alpha}}\int_{z'\in K_{U^{-1}\alpha'}}\frac{2^{2d\left(\alpha\wedge\alpha'\right)}}{2^{d^*([\alpha]\wedge[\alpha'])}} f(z')\,d\mu(z')\,f(z)\,d\mu(z)\,dU.$$

Now let $(fd\mu)_U = (fd\mu)\circ U^{-1}$ for each $U \in \mathcal{U}_n$, so that $f(z')\,d\mu(z') = (fd\mu)_U(Uz')$. Then if we make the change of variable $w' = Uz'$ and $w = Uz$ in the inner integrals above, II becomes

$$\begin{aligned}
&C\int_{\mathcal{U}_n}\left\{\sum_{\alpha,\alpha'\in\mathcal{T}_n}\int_{w\in K_\alpha}\int_{w'\in K_{\alpha'}}\frac{2^{2d\left(\alpha\wedge\alpha'\right)}}{2^{d^*([\alpha]\wedge[\alpha'])}}(fd\mu)_U(w')\,(fd\mu)_U(w)\right\}dU\\
&\quad= C\int_{\mathcal{U}_n}\left\{\sum_{\alpha,\alpha'\in\mathcal{T}_n}\frac{2^{2d\left(\alpha\wedge\alpha'\right)}}{2^{d^*([\alpha]\wedge[\alpha'])}}(fd\mu)_U(\alpha')\,(fd\mu)_U(\alpha)\right\}dU.
\end{aligned}$$

Now we write

$$(fd\mu)_U(\alpha) = \int_{U^{-1}K_\alpha} fd\mu = \left(\frac{1}{\left|U^{-1}K_\alpha\right|_\mu}\int_{U^{-1}K_\alpha} fd\mu\right)\mu\left(U^{-1}\alpha\right),$$

so that we obtain an estimate for II from (76) as follows:

$$
\begin{aligned}
II &\leq C \int_{\mathcal{U}_n} \left\{ \sum_{\alpha,\alpha' \in \mathcal{T}_n} \frac{2^{2d(\alpha \wedge \alpha')}}{2^{d^*([\alpha] \wedge [\alpha'])}} \left(f d\mu\right)_U (\alpha') \left(f d\mu\right)_U (\alpha) \right\} dU \qquad (90) \\
&\leq C \int_{\mathcal{U}_n} \left\{ \sum_{\alpha,\alpha' \in \mathcal{T}_n} \frac{2^{2d(\alpha \wedge \alpha')}}{2^{d^*([\alpha] \wedge [\alpha'])}} \left(\frac{1}{|U^{-1} K_{\alpha'}|_\mu} \int_{U^{-1} K_{\alpha'}} f d\mu \right) \mu\left(U^{-1} \alpha'\right) \right. \\
&\quad \left. \times \left(\frac{1}{|U^{-1} K_\alpha|_\mu} \int_{U^{-1} K_\alpha} f d\mu \right) \mu\left(U^{-1} \alpha\right) \right\} dU \\
&\leq C \int_{\mathcal{U}_n} \left\{ \sum_{\alpha \in \mathcal{T}_n} \left(\frac{1}{|U^{-1} K_\alpha|_\mu} \int_{U^{-1} K_\alpha} f d\mu \right)^2 \mu\left(U^{-1} \alpha\right) \right\} dU \\
&\leq C \int_{\mathcal{U}_n} \left\{ \sum_{\alpha \in \mathcal{T}_n} \int_{U^{-1} K_\alpha} f^2 d\mu \right\} dU = C \left\| f \right\|_{L^2(\widetilde{\mu})}^2 .
\end{aligned}
$$

Combining the estimates (89) and (90) for terms I and II, we thus obtain the bilinear inequality (68) when $f = g$, and this suffices for the general inequality. This completes the proof of the equivalence of (68) and (76).

Now (76) can be rewritten as

$$
\sum_{\alpha \in \mathcal{T}_n} f(\alpha) \left\{ T_\mu g(\alpha) \right\} \mu(\alpha) \leq C \left\| f \right\|_{\ell^2(\mu)} \left\| g \right\|_{\ell^2(\mu)} , \qquad (91)
$$

for all $f, g \geq 0$ on $\mathcal{T}_n$, and where T_μ is given in (74):

$$
T_\mu g(\alpha) = \sum_{\alpha' \in \mathcal{T}_n} 2^{2d(\alpha \wedge \alpha') - d^*\left([\alpha] \wedge [\alpha']\right)} g(\alpha') \mu(\alpha') .
$$

Upon using the Cauchy-Schwartz inequality and taking the supremum over all f with $\left\| f \right\|_{\ell^2(\mu)} = 1$ in (91), we obtain the equivalence of (91) and the discrete inequality (73), where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.

4.2.3 Carleson measures for H_n^2 and inequalities for positive quantities

Using Propositions 27 and 29, we can characterize Carleson measures for the Drury-Arveson Hardy space H_n^2 by either (68) or (73). Recall that $\widehat{\mu}(\alpha) = \mu(\alpha) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}_n$.

Theorem 30 *Let μ be a positive measure on the ball $\mathbb{B}_n$ with n finite. Then the following conditions are equivalent:*

1. *μ is a Carleson measure on the Drury-Arveson space H_n^2,*

2. μ satisfies (68),

3. $\widehat{\mu}$ satisfies (73) for all unitary rotations of a fixed Bergman tree.

In Proposition 33 of the next subsubsection, we will complete the characterization of Carleson measures for the Drury-Arveson space by giving necessary and sufficient conditions for (73) taken over all unitary rotations of a fixed Bergman tree, namely the split tree condition (107) and the simple condition (92), both given below, taken over all unitary rotations of a fixed Bergman tree.. We record here the necessity of the simple condition.

Lemma 31 *If μ is a Carleson measure on the Drury-Arveson space H_n^2, then μ satisfies the simple condition,*

$$2^{d(\alpha)} I^* \mu(\alpha) \leq C, \quad \alpha \in \mathcal{T}_n. \tag{92}$$

Recall that $p\sigma = 1$ and that $\theta = \frac{\ln 2}{2}$ so that $1 - |w|^2 \approx 2^{-d(\alpha)} = 2^{-p\sigma d(\alpha)}$ for $w \in K_\alpha$.

Proof. (of the lemma) In fact the analogous statement holds for all $\sigma > 0$. Recall from Subsubsection 2.3.2 that $B_2^\sigma(\mathbb{B}_n)$ can be realized as $\mathcal{H}_k$ with kernel function $k(w,z) = \left(\frac{1}{1-\overline{w}\cdot z}\right)^{2\sigma}$ on $\mathbb{B}_n$. This function satisfies

$$\left\| \left(\frac{1}{1-\overline{w}\cdot z}\right)^{2\sigma} \right\|_{B_2^\sigma(\mathbb{B}_n)}^2 = \left(\frac{1}{1-|w|^2}\right)^{2\sigma}.$$

Testing the Carleson embedding on these functions quickly leads to the desired estimates.

Later we will use the fact that the condition SC(1/2) is sufficient to insure the tree condition with $\sigma < 1/2$. Rather than prove that in isolation we take the opportunity to record two strengthenings of the condition SC(σ) each of which is sufficient to imply the corresponding tree condition (3). Either of the two suffices to establish that, given any $\varepsilon > 0$, the condition SC($\sigma + \varepsilon$) implies (3).

For $\sigma > 0$. We will say that a measure μ satisfies the strengthened simple condition if there is a *summable* function $h(\cdot)$ such that

$$2^{2\sigma d(\alpha)} I^* \mu(\alpha) \leq C h(d(\alpha)), \quad \alpha \in \mathcal{T}_n, \tag{93}$$

For $0 < p < 1$ we say that μ satisfies the ℓ^p-simple condition if

$$2^{2\sigma d(\alpha)} \left(\sum_{\beta \geq \alpha} \mu(\beta)^p \right)^{\frac{1}{p}} \leq C, \quad \alpha \in \mathcal{T}_n. \tag{94}$$

Note that the choices $h \equiv 1$ and $p = 1$ recapture the simple condition SC(σ): $2^{2\sigma d(\alpha)} I^* \mu(\alpha) \leq C$.

Lemma 32 *Let $\sigma > 0$. If μ satisfies* either *the strengthened simple condition (93),* or *the ℓ^p-simple condition (94) for some $0 < p < 1$, then μ satisfies the tree condition (3).*

For the particular case when μ is the interpolation measure associated with a separated sequence of points in the unit disk the result about the ℓ^p-simple condition is Theorem 4 on page 38 of [34].

Proof. The left side of (3) satisfies

$$\begin{aligned}\sum_{\gamma\geq\alpha} 2^{2\sigma d(\gamma)} I^*\mu\left(\gamma\right)^2 &= \sum_{\delta,\delta'\geq\alpha}\sum_{\alpha\leq\gamma\leq\delta\wedge\delta'} 2^{2\sigma d(\gamma)}\mu\left(\delta\right)\mu\left(\delta'\right)\\ &\leq C\sum_{\delta,\delta'\geq\alpha} 2^{2\sigma d\left(\delta\wedge\delta'\right)}\mu\left(\delta\right)\mu\left(\delta'\right).\end{aligned}$$

If (93) holds, then we continue with

$$\begin{aligned}\sum_{\delta,\delta'\geq\alpha} 2^{2\sigma d\left(\delta\wedge\delta'\right)}\mu\left(\delta\right)\mu\left(\delta'\right) &\leq \sum_{\delta\geq\alpha}\mu\left(\delta\right)\sum_{\alpha\leq\beta\leq\delta} 2^{2\sigma d(\beta)} I^*\mu\left(\beta\right)\\ &\leq C\sum_{\delta\geq\alpha}\mu\left(\delta\right)\sum_{\alpha\leq\beta\leq\delta} h\left(d\left(\beta\right)\right)\\ &\leq C\sum_{\delta\geq\alpha}\mu\left(\delta\right) = CI^*\mu\left(\alpha\right),\end{aligned}$$

which yields (3). On the other hand, if (94) holds with $0 < p < 1$, then we use

$$\mu\left(\delta\right)\mu\left(\delta'\right) \leq \mu\left(\delta\right)^{2-p}\mu\left(\delta'\right)^p + \mu\left(\delta'\right)^{2-p}\mu\left(\delta\right)^p,$$

together with the symmetry in δ and δ', to continue with

$$\begin{aligned}\sum_{\delta,\delta'\geq\alpha} 2^{2\sigma d\left(\delta\wedge\delta'\right)}\mu\left(\delta\right)^{2-p}\mu\left(\delta'\right)^p &= \sum_{\delta\geq\alpha}\mu\left(\delta\right)^{2-p}\sum_{\alpha\leq\beta\leq\delta} 2^{2\sigma d(\beta)}\left(\sum_{\delta'\geq\beta}\mu\left(\delta'\right)^p\right)\\ &\leq C\sum_{\delta\geq\alpha}\mu\left(\delta\right)^{2-p}\sum_{\alpha\leq\beta\leq\delta} 2^{2\sigma(1-p)d(\beta)}\\ &\leq C_p\sum_{\delta\geq\alpha}\mu\left(\delta\right)^{2-p} 2^{2\sigma(1-p)d(\delta)}\\ &\leq C_p\sum_{\delta\geq\alpha}\mu\left(\delta\right) = C_pI^*\mu\left(\alpha\right),\end{aligned}$$

which again yields (3). The final inequality here follows since

$$\mu\left(\delta\right)^{1-p} 2^{2\sigma(1-p)d(\delta)} = \left(\mu\left(\delta\right) 2^{2\sigma d(\delta)}\right)^{1-p} \leq \left(I^*\mu\left(\delta\right) 2^{2\sigma d(\delta)}\right)^{1-p} \leq C$$

by the usual simple condition, an obvious consequence of (94) when $0 < p < 1$.

The two conditions in the lemma are independent of each other. We offer ingredients for the examples that show this but omit the details of the verification. Suppose $\sigma = 1/2$ and let $\mathcal{T}_0$ be the linear tree. The measure $\mu(\alpha) = 2^{-d(\alpha)}$ satisfies (94) for any $p > 0$ but fails (93) for any summable h. Now consider the binary tree $\mathcal{T}_1$. Set $\mu_N(\alpha) = 2^{-N} N^{-1} (\log N)^{-2}$. With the choice $h(n) = n^{-1}(\log n)^{-2}, n \geq 2$, the measures μ_N satisfy (93) uniformly in N. However with the choice of $\alpha = o$ the left side of (94) is $2^{-N+N/p} N^{-1} (\log N)^{-2}$ which is unbounded in N for any fixed $p < 1$.

4.2.4 The split tree condition

The bilinear inequality associated with (73) is

$$\sum_{\alpha \in \mathcal{T}_n} f(\alpha) T_\mu g(\alpha) \mu(\alpha) = \sum_{\alpha,\beta \in \mathcal{T}_n} 2^{2d(\alpha\wedge\beta) - d^*([\alpha]\wedge[\beta])} f\mu(\alpha) g\mu(\beta) \tag{95}$$
$$\leq C \left(\sum_{\alpha \in \mathcal{T}_n} f(\alpha)^2 \mu(\alpha) \right)^{\frac{1}{2}} \left(\sum_{\alpha \in \mathcal{T}_n} g(\alpha)^2 \mu(\alpha) \right)^{\frac{1}{2}}.$$

By Theorem 30 and Lemma 31, the simple condition (6) is necessary for (95). We now derive another necessary condition for (95) to hold, namely the split tree condition (7). First we set $f = g = \chi_{S(\eta)}$ in (95) to obtain

$$\sum_{\alpha,\beta \geq \eta} 2^{2d(\alpha\wedge\beta) - d^*([\alpha]\wedge[\beta])} \mu(\alpha) \mu(\beta) \leq C I^* \mu(\eta), \qquad \eta \in \mathcal{T}_n.$$

If we organize the sum on the left hand side by summing first over rings, we obtain

$$\sum_{A,B \in \mathcal{R}_n} \sum_{\substack{\alpha,\beta \geq \eta \\ \alpha \in A, \beta \in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d^*(A\wedge B)}} \mu(\alpha) \mu(\beta) = \sum_{C \in \mathcal{R}_n} \sum_{\substack{A,B \in \mathcal{R}_n \\ A\wedge B = C}} \sum_{\substack{\alpha,\beta \geq \eta \\ \alpha \in A, \beta \in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d^*(A\wedge B)}} \mu(\alpha) \mu(\beta).$$

Now define $A \curlywedge B = C$ to mean the more restrictive condition that both $A \wedge B = C$ and $d^*(A \wedge B) - d(C)$ is bounded (thus requiring that $A \wedge B$ is not "artificially" too much closer to the root than it ought to be due to the vagaries of the particular tree structure). We can then restrict the sum over A and B above to $A \curlywedge B = C$ which permits $2^{d^*(A\wedge B)}$ to be replaced by $2^{d(C)}$. The result is

$$\sum_{C \in \mathcal{R}_n} \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} \sum_{\substack{\alpha,\beta \geq \eta \\ \alpha \in A, \beta \in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d(C)}} \mu(\alpha) \mu(\beta) \leq C I^* \mu(\eta), \qquad \eta \in \mathcal{T}_n.$$

This is the split tree condition on the Bergman tree $\mathcal{T}_n$, which dominates the more transparent form

$$\sum_{k \geq 0} \sum_{\gamma \geq \eta} 2^{d(\gamma) - k} \sum_{(\delta,\delta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\delta) I^* \mu(\delta') \leq C I^* \mu(\eta), \qquad \eta \in \mathcal{T}_n,$$

with $\gamma = \alpha \wedge \beta$ and $k = d(c) - d(\gamma)$, where as in Definition 1, the set $\mathcal{G}^{(k)}(\gamma)$ consists of pairs (δ, δ') of grand^k-children of γ in $\mathcal{G}^{(k)}(\gamma) \times \mathcal{G}^{(k)}(\gamma)$ which satisfy $\delta \wedge \delta' = \gamma$, $\left[A^2\delta\right] = \left[A^2\delta'\right]$ (which implies $d([\delta], [\delta']) \leq 4$) and $d^*([\delta], [\delta']) = 4$. Note that $\mathcal{G}^{(0)}(\gamma) = \mathcal{G}(\gamma)$ is the set of grandchildren of γ.

To show the sufficiency of the simple condition (6) and the split tree condition (7) taken over all unitary rotations of a fixed Bergman tree, we begin by claiming that the left hand side of (95) satisfies

$$\sum_{A,B\in\mathcal{R}_n} \sum_{\substack{\alpha\in A\\ \beta\in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d^*(A\wedge B)}} f\mu(\alpha)\, g\mu(\beta)$$

$$\leq C \int_{\mathcal{U}_n} \sum_{C\in U^{-1}\mathcal{R}_n} \sum_{\substack{A,B\in U^{-1}\mathcal{R}_n\\ A\curlywedge B=C}} \sum_{\substack{\alpha\in A\\ \beta\in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d(C)}} f\mu(\alpha)\, g\mu(\beta)\ dU, \quad (96)$$

To see this we note that from (78) and (79), we have

$$d^*([z] \wedge [z']) \leq d([Uz] \wedge [Uz']) + C,$$

for $U \in \Sigma$ where $|\Sigma| \geq c > 0$. Moreover, this inequality persists in the following somewhat stronger form: for any fixed rings A, B associated to the tree $\mathcal{R}_n$, there is Σ with $|\Sigma| \geq c > 0$ such that for any $U \in \Sigma$, if $A', B' \in U^{-1}\mathcal{R}_n$ satisfy $A \cap A' \neq \phi, B \cap B' \neq \phi$, then $d(A' \wedge B') - d^*(A \wedge B)$ is bounded and hence $A' \curlywedge B' = A' \wedge B'$. Thus

$$\sum_{\substack{\alpha\in A\\ \beta\in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d^*(A\wedge B)}} f\mu(\alpha)\, g\mu(\beta)$$

$$\leq C \int_{\mathcal{U}_n} \sum_{\substack{C'\in U^{-1}\mathcal{R}_n\\ A'\curlywedge B'=C'\\ A\cap A', B\cap B'\neq\phi}} \sum_{\alpha\in A', \beta\in B'} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d(C')}} f\mu(\alpha)\, g\mu(\beta)\ dU$$

as required. Thus it will suffice to prove that (6) and the split tree condition (7) for the tree $U^{-1}\mathcal{T}_n$ imply

$$\sum_{C\in U^{-1}\mathcal{R}_n} \sum_{\substack{A,B\in U^{-1}\mathcal{R}_n\\ A\curlywedge B=C}} \sum_{\substack{\alpha\in A\\ \beta\in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d(C)}} f\mu(\alpha)\, g\mu(\beta) \quad (97)$$

$$\leq C \left(\sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2 \mu(\alpha) \right)^{\frac{1}{2}} \left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)^2 \mu(\alpha) \right)^{\frac{1}{2}},$$

with a constant C independent of $U \in \mathcal{U}_n$. Without loss of generality we prove (97) when U is the identity.

Define the projection P_C from functions $h = \{h(\alpha)\}_{\alpha \in A}$ on the ring A to functions $P_C h$ on the ring C (provided $C \leq A$) by

$$P_C h = \left\{ \sum_{\substack{\alpha \in A \\ \alpha \geq \gamma}} h(\alpha) \right\}_{\gamma \in C}.$$

We also define the "Poisson kernel" $\mathbb{P}_C$ at scale C to be the mapping taking functions $h = \{h(\gamma')\}_{\gamma' \in C}$ on C to functions $\mathbb{P}_C h = \{\mathbb{P}_C h(\gamma)\}_{\gamma \in C}$ on C given by

$$\mathbb{P}_C h = \left\{ \sum_{\gamma' \in A} \frac{2^{2d(\gamma \wedge \gamma')}}{2^{d(C)}} h(\gamma') \right\}_{\gamma \in C}.$$

Now if f_A denotes the restriction $\chi_A f$ of f to the ring A, we can write the left side of (97) as approximately

$$\begin{aligned} &\sum_{C \in \mathcal{R}_n} \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} \sum_{\gamma \in C} \mathbb{P}_C\left(P_C\left(f_A \mu\right)\right)(\gamma) P_C\left(g_B \mu\right)(\gamma) \\ &\quad = \sum_{C \in \mathcal{R}_n} \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} \left\langle \mathbb{P}_C\left(P_C\left(f_A \mu\right)\right), P_C\left(g_B \mu\right)\right\rangle_C, \end{aligned}$$

where the inner product $\langle F, G \rangle_C$ is given by $\sum_{\gamma \in C} F(\gamma) G(\gamma)$. At this point we notice that the Poisson kernel

$$\mathbb{P}_C(\gamma, \gamma') = \frac{2^{2d(\gamma \wedge \gamma')}}{2^{d(C)}}$$

is a geometric sum of averaging operators $\mathbb{A}_C^k$ with kernel

$$\mathbb{A}_C^k(\gamma, \gamma') = 2^{d(C)-k} \chi_{\{d(\gamma \wedge \gamma') = d(C)-k\}},$$

namely

$$\mathbb{P}_C(\gamma, \gamma') = \sum_{k=0}^{d(C)} 2^{-k} \mathbb{A}_C^k(\gamma, \gamma'). \tag{98}$$

We now consider the bilinear inequality with $\mathbb{P}_C$ replaced by $\mathbb{A}_C^0$:

$$\sum_{C \in \mathcal{R}_n} \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} \left\langle \mathbb{A}_C^0\left(P_C\left(f_A \mu\right)\right), P_C\left(g_B \mu\right)\right\rangle_C \leq C \|f\|_{\ell^2(\mu)} \|g\|_{\ell^2(\mu)}. \tag{99}$$

The left side of (99) is

$$\begin{aligned} &\sum_{C \in \mathcal{R}_n} 2^{d(C)} \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} \left\langle P_C\left(f_A \mu\right), P_C\left(g_B \mu\right)\right\rangle_C \\ &\quad = \sum_{C \in \mathcal{R}_n} 2^{d(C)} \sum_{\gamma \in C} \left\{ \sum_{\substack{A,B \in \mathcal{R}_n \\ A \curlywedge B = C}} I^*\left(f_A \mu\right)(\gamma) I^*\left(g_B \mu\right)(\gamma) \right\}. \end{aligned}$$

For fixed $\gamma \in C$, we dominate the sum $\sum_{A,B\in\mathcal{R}_n:A\curlywedge B=C}$ in braces above by

$$\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}} I^*\left(f_A\mu\right)(\gamma)\, I^*\left(g_B\mu\right)(\gamma) \leq I^*\left(f\mu\right)(\gamma)\left(g\mu\right)(\gamma)$$
$$+ \left(f\mu\right)(\gamma)\, I^*\left(g\mu\right)(\gamma) + \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],\left[\delta'\right]\right)=4}} I^*\left(f\mu\right)(\delta)\, I^*\left(g\mu\right)(\delta')\,.$$

The first two terms easily satisfy the bilinear inequality using only the simple condition (92). Indeed,

$$\begin{aligned}\sum_{C\in\mathcal{R}_n} 2^{d(C)} \sum_{\gamma\in C} I^*\left(f\mu\right)(\gamma)\left(g\mu\right)(\gamma) &= \sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} I^*\left(f\mu\right)(\gamma)\left(g\mu\right)(\gamma)\\ &= \sum_{\gamma\in\mathcal{T}_n} I\left(2^d f\mu\right)(\gamma)\, g\left(\gamma\right)\mu\left(\gamma\right)\\ &\leq \left\|I\left(2^d f\mu\right)\right\|_{\ell^2(\mu)} \left\|g\right\|_{\ell^2(\mu)}.\end{aligned}$$

Now the inequality $\left\|I\left(2^d f\mu\right)\right\|_{\ell^2(\mu)} \leq C\left\|f\right\|_{\ell^2(\mu)}$ can be rewritten

$$\sum_{\gamma\in\mathcal{T}_n} Ih\left(\gamma\right)^2 \mu\left(\gamma\right) \leq C \sum_{\gamma\in\mathcal{T}_n} h\left(\gamma\right)^2 2^{-2d(\gamma)}\mu\left(\gamma\right),$$

which by Theorem 19, is equivalent to the condition

$$\sum_{\gamma\geq\alpha} I^*\mu\left(\gamma\right)^2 2^{2d(\gamma)}\mu\left(\gamma\right) \leq CI^*\mu\left(\alpha\right), \qquad \alpha\in\mathcal{T}_n,$$

which in turn is trivially implied by the simple condition $I^*\mu\left(\gamma\right)^2 2^{2d(\gamma)} \leq C^2$.

It remains then to consider the "split" bilinear inequality

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],\left[\delta'\right]\right)=4}} I^*\left(f\mu\right)(\delta)\, I^*\left(g\mu\right)(\delta') \leq C\left\|f\right\|_{\ell^2(\mu)}\left\|g\right\|_{\ell^2(\mu)}, \tag{100}$$

or equivalently the corresponding quadratic inequality obtained by setting $f = g$:

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],\left[\delta'\right]\right)=4}} I^*\left(f\mu\right)(\delta)\, I^*\left(f\mu\right)(\delta') \leq C\sum_{\alpha\in\mathcal{T}_n} f\left(\alpha\right)^2\mu\left(\alpha\right). \tag{101}$$

Note that the restriction to $k = 0$ in the split tree condition (7) yields the following necessary condition for (101):

$$\sum_{\gamma\geq\alpha} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],\left[\delta'\right]\right)=4}} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right) \leq CI^*\mu\left(\alpha\right), \qquad \alpha\in\mathcal{T}_n. \tag{102}$$

We now show that (102) and (92) together imply (101). To see this write the left side of (101) as

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],[\delta']\right)=4}} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right) \frac{I^*\left(f\mu\right)\left(\delta\right) I^*\left(f\mu\right)\left(\delta'\right)}{I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right)},$$

and using the symmetry in δ,δ' we bound it by

$$\begin{aligned}&\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],[\delta']\right)=4}} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right) \left(\frac{I^*\left(f\mu\right)\left(\delta\right)}{I^*\mu\left(\delta\right)}\right)^2\\ &\quad= \sum_{\delta\in\mathcal{T}_n} \left(\frac{I^*\left(f\mu\right)\left(\delta\right)}{I^*\mu\left(\delta\right)}\right)^2 \sum_{\substack{\delta'\in\mathcal{G}\left(A^2\delta\right)\\ d^*\left([\delta],[\delta']\right)=4}} 2^{d\left(A^2\delta\right)} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right).\end{aligned}$$

By Theorem 21, this last term is dominated by the right side of (101) provided $I^*\sigma\left(\alpha\right)\leq CI^*\mu\left(\alpha\right)$ for all $\alpha\in\mathcal{T}_n$ where $\sigma\left(\delta\right)$ is given by

$$\sigma\left(\delta\right) = \sum_{\substack{\delta'\in\mathcal{G}\left(A^2\delta\right)\\ d^*\left([\delta],[\delta']\right)=4}} 2^{d\left(A^2\delta\right)} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right).$$

This latter condition can be expressed as

$$\begin{aligned}&\sum_{\gamma\geq\alpha} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{G}(\gamma)\\ d^*\left([\delta],[\delta']\right)=4}} I^*\mu\left(\delta\right) I^*\mu\left(\delta'\right) + 2^{d(A\alpha)} \sum_{\substack{\delta'\in\mathcal{G}(A\delta)\\ d^*\left([\delta],[\delta']\right)=4}} I^*\mu\left(A\alpha\right) I^*\mu\left(\delta'\right)\\ &\quad+ 2^{d\left(A^2\alpha\right)} \sum_{\substack{\delta'\in\mathcal{G}\left(A^2\delta\right)\\ d^*\left([\delta],[\delta']\right)=4}} I^*\mu\left(A^2\alpha\right) I^*\mu\left(\delta'\right) \leq CI^*\mu\left(\alpha\right), \qquad \alpha\in\mathcal{T}_n.\end{aligned} \tag{103}$$

Now the necessary condition (102) shows that the first sum in (103) is at most $CI^*\mu\left(\alpha\right)$, while the simple condition (92) yields $2^{d(A\alpha)}I^*\mu\left(\delta'\right)\leq C$, which shows that the second sum in (103) is at most $CI^*\mu\left(\alpha\right)$. The third sum is handled similarly and this completes the proof that (99) holds when both (102) and (92) hold.

To handle the averaging operators $\mathbb{A}_C^k$ for $k>0$, we compute that for $D\in$

$\mathcal{R}_n$,

$$\sum_{C\in\mathcal{C}^{(k-1)}(D)}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}\left\langle \mathbb{A}_C^k\left(P_C\left(f_A\mu\right)\right),P_C\left(g_B\mu\right)\right\rangle_C$$
$$=\sum_{C\in\mathcal{C}^{(k-1)}(D)}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}2^{d(C)-k}\sum_{\substack{\gamma,\gamma'\in C\\ d\left(\gamma\wedge\gamma'\right)=d(D)}}P_C\left(f_A\mu\right)(\gamma)\,P_C\left(g_B\mu\right)(\gamma')$$
$$=\sum_{C\in\mathcal{C}^{(k-1)}(D)}2^{d(C)-k}\left(\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta\\ \gamma'\geq\delta'}}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}P_C\left(f_A\mu\right)(\gamma)\,P_C\left(g_B\mu\right)(\gamma')\right).$$

Summing this over all rings $D\in\mathcal{R}_n$, and then summing in $k>0$, we obtain

$$\sum_{C\in\mathcal{R}_n}\sum_{A,B\in\mathcal{R}_n:A\curlywedge B=C}\left\langle\sum_{k>0}2^{-k}\mathbb{A}_C^k\left(P_C\left(f_A\mu\right)\right),P_C\left(g_B\mu\right)\right\rangle_C$$
$$=\sum_{k>0}2^{-k}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}\left\langle\mathbb{A}_C^k\left(P_C\left(f_A\mu\right)\right),P_C\left(g_B\mu\right)\right\rangle_C$$
$$=\sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)}2^{d(C)-2k}\left(*\right).$$

The term $(*)$ is

$$(*)=\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta,\gamma'\geq\delta'}}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}P_C\left(f_A\mu\right)(\gamma)\,P_C\left(g_B\mu\right)(\gamma')$$

and it satisfies $(*)\leq I+II+III$ with

$$I=\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta,\gamma'\geq\delta'}}\left(f\mu\right)(\gamma)\left(\sum_{B\geq C}P_C\left(g_B\mu\right)(\gamma')\right)$$

$$II=\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta,\gamma'\geq\delta'}}\left(\sum_{A\geq C}P_C\left(f_A\mu\right)(\gamma)\right)\left(g\mu\right)(\gamma')$$

$$III=\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta,\gamma'\geq\delta'}}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}P_C\left(f_A\mu\right)(\gamma)\,P_C\left(g_B\mu\right)(\gamma').$$

We now analyze these sums in terms of the operator I^*. The first two, I and II, are are similar to each other and can be controlled by the simple condition (92) alone. Indeed, $\sum_{B\geq C} P_C\left(g_B\mu\right)\left(\gamma'\right) = I^*\mu\left(\gamma'\right)$ and

$$\sum_{k>0}\sum_{D\in\mathcal{R}_n}\left\{\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\left(\sum_{\substack{\delta,\delta'\in D\\ d\left(\delta,\delta'\right)=2}}\sum_{\substack{\gamma,\gamma'\in C\\ \gamma\geq\delta,\gamma'\geq\delta'}}\left(f\mu\right)\left(\gamma\right)I^*\mu\left(\gamma'\right)\right)\right\}$$

has bilinear kernel function $K\left(\gamma,\beta\right) = 2^{d(\gamma)-2k}$ where $k = d\left(\gamma\right) - d\left(\gamma\wedge\gamma'\right)$ and γ' is the unique element of the ring $C = [\gamma]$ with $\beta\geq\gamma'$. Since $d\left(\gamma\wedge\beta\right) = d\left(\gamma\wedge\gamma'\right)$, we thus have

$$K\left(\gamma,\beta\right) = 2^{d(\gamma)-2\left(d(\gamma)-d\left(\gamma\wedge\gamma'\right)\right)} = 2^{2d(\gamma\wedge\beta)-\min\{d(\gamma),d(\beta)\}},$$

and the case $r = 1$ of Theorem 36 below shows that this kernel is controlled by the simple condition. We now turn to III. Using $(+)$ to denote a set of summation indices:

$$(+) = \left\{\eta,\eta' : A^2\eta, A^2\eta'\in C, d\left(\eta\wedge\eta'\right) = d\left(D\right)+1, d^*\left([\eta]\wedge[\eta']\right) = 4\right\}$$

we can rewrite III as

$$III = \sum_{(+)} I^*\left(f\mu\right)\left(\eta\right)I^*\left(g\mu\right)\left(\eta'\right).$$

Setting $f = g$ we see we must show that

$$\sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\sum_{(+)} I^*\left(f\mu\right)\left(\eta\right)I^*\left(f\mu\right)\left(\eta'\right)\leq C\left\|f\right\|^2_{\ell^2(\mu)}. \quad (104)$$

Just as in handling the bilinear inequality for $\mathbb{A}^0_C$ above, we exploit the symmetry in η,η' to obtain

$$\begin{aligned}
&\sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\sum_{(+)} I^*\left(f\mu\right)\left(\eta\right)I^*\left(f\mu\right)\left(\eta'\right)\\
&\quad = \sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\sum_{(+)}\left[I^*\mu\left(\eta\right)I^*\mu\left(\eta'\right)\right]\frac{I^*\left(f\mu\right)\left(\eta\right)I^*\left(f\mu\right)\left(\eta'\right)}{I^*\mu\left(\eta\right)I^*\mu\left(\eta'\right)}\\
&\quad \leq \sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\sum_{(+)} I^*\mu\left(\eta\right)I^*\mu\left(\eta'\right)\left(\frac{I^*\left(f\mu\right)\left(\eta\right)}{I^*\mu\left(\eta\right)}\right)^2.
\end{aligned}$$

Now we apply Theorem 21 to obtain that the last expression above is dominated by the right side of (104) provided we have the condition, for $\alpha\in\mathcal{T}_n$.

$$\sum_{\eta:\eta\geq\alpha}\left\{\sum_{k>0}\sum_{D\in\mathcal{R}_n}\sum_{C\in\mathcal{C}^{(k-1)}(D)} 2^{d(C)-2k}\sum_{(+)} I^*\mu\left(\eta\right)I^*\mu\left(\eta'\right)\right\}\leq CI^*\mu\left(\alpha\right) \quad (105)$$

As before, this condition is implied by the simple condition (92) together with the restriction to $k > 0$ in the split tree condition (7):

$$\sum_{k>0} \sum_{\gamma \geq \alpha} 2^{d(\gamma)-k} \sum_{\eta,\eta' \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\eta) I^* \mu(\eta') \leq C I^* \mu(\alpha), \qquad \alpha \in \mathcal{T}_n, \tag{106}$$

where we recall the notation from Definition 1,

$$\mathcal{G}^{(k)}(\gamma) = \left\{ (\eta, \eta') \in \mathcal{G}^{(k)}(\gamma) \times \mathcal{G}^{(k)}(\gamma) : \begin{array}{c} \eta \wedge \eta' = \gamma, \left[A^2 \eta\right] = \left[A^2 \eta'\right] \\ d^*([\eta], [\eta']) \geq 2 \end{array} \right\}.$$

We now show that (105) is implied by the simple condition (92) together with (106). The proof is analogous to the argument used to establish that (92) and (102) imply (103) above. We rewrite the left side of (105) as

$$\sum_{k>0} \sum_{\gamma \geq \alpha} 2^{d(\gamma)-k} \sum_{(\eta,\eta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\eta) I^* \mu(\eta') + \mathsf{REST}.$$

Now the terms in REST that have $\eta = \alpha$ are dominated by

$$\begin{aligned} &\sum_{k>0} 2^{d(\alpha)-2k} I^* \mu(\alpha) \sum_{\substack{\eta' \in [\eta] \\ d(\eta \wedge \eta') = d(\alpha)-k}} I^* \mu(\eta') \\ &\qquad \leq \sum_{k>0} 2^{d(\alpha)-2k} I^* \mu(\alpha) \sum_{\substack{\eta' \in [\eta] \\ d(\eta \wedge \eta') = d(\alpha)-k}} C 2^{-d(\alpha)} \\ &\qquad \leq C \sum_{k>0} 2^{-k} I^* \mu(\alpha) \leq C I^* \mu(\alpha), \end{aligned}$$

as required. However, we must also sum over the terms having simultaneously $\eta > \alpha$ and not $\eta' > \alpha$. We organize this sum by summing over the pairs $(\eta, \eta') \in \mathcal{G}^{(k)}(\gamma)$ for which $\eta \wedge \eta'$ equals a given $\gamma \in [o, A\alpha]$, and then splitting this sum over those $\eta \in \mathcal{C}^{(\ell)}(\alpha)$, $\ell > 0$, so that $d(\alpha) + \ell = d(\eta) = d(\gamma) + k$, and obtain the following:

$$\begin{aligned} &\sum_{\gamma < \alpha} \sum_{\ell > 0} \sum_{\eta \in \mathcal{C}^{(\ell)}(\alpha)} \sum_{\substack{\eta' : d(\eta') = d(\eta) \\ \eta \wedge \eta' = \gamma}} 2^{d(\eta) - 2[d(\alpha) + \ell - d(\gamma)]} I^* \mu(\eta) I^* \mu(\eta') \\ &= \sum_{\gamma < \alpha} 2^{2d(\gamma) - d(\alpha)} \sum_{\ell > 0} 2^{-\ell} \sum_{\eta \in \mathcal{C}^{(\ell)}(\alpha)} \sum_{\substack{\eta' : d(\eta') = d(\eta) \\ \eta \wedge \eta' = \gamma}} I^* \mu(\eta) I^* \mu(\eta') \\ &\leq \sum_{\gamma < \alpha} 2^{2d(\gamma) - d(\alpha)} \sum_{\ell > 0} 2^{-\ell} I^* \mu(\alpha) I^* \mu(\gamma) \\ &\leq \left\{ C \sum_{\gamma < \alpha} 2^{d(\gamma) - d(\alpha)} \right\} I^* \mu(\alpha) = C I^* \mu(\alpha). \end{aligned}$$

Combining the above with (98) and (99) we obtain

$$\sum_{C\in\mathcal{R}_n}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\curlywedge B=C}}\left\langle \mathbb{P}_C\left(P_C\left(f_A\mu\right)\right),P_C\left(g_B\mu\right)\right\rangle_C \leq C\left\|f\right\|_{\ell^2(\mu)}\left\|g\right\|_{\ell^2(\mu)},$$

and hence (95), provided that (92), (102) and (106) all hold. We have thus obtained the following characterization of (73) taken over all unitary rotations of a fixed Bergman tree.

Proposition 33 *A positive measure μ on $\mathbb{B}_n$ satisfies (73), where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree, if and only if μ satisfies the simple condition (92) and the following split tree condition,*

$$\sum_{k\geq 0}\sum_{\gamma\geq\alpha}2^{d(\gamma)-k}\sum_{(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)}I^*\mu\left(\delta\right)I^*\mu\left(\delta'\right)\leq CI^*\mu\left(\alpha\right),\qquad \alpha\in\mathcal{T}_n, \tag{107}$$

taken over all unitary rotations of the same Bergman tree, and where $\mathcal{G}^{(k)}$ is given by Definition 1. Moreover,

$$\begin{aligned} c_n\sup_{\mathcal{T}_n}\left\|\mu\right\|_{Carleson(\mathcal{T}_n)} &\leq \sup_{\alpha\in\mathcal{T}_n}\sqrt{2^{d(\alpha)}I^*\mu\left(\alpha\right)}\\ &\quad+\sup_{\substack{\alpha\in\mathcal{T}_n\\ I^*\mu(\alpha)>0}}\sqrt{\frac{1}{I^*\mu\left(\alpha\right)}\sum_{k\geq 0}\sum_{\gamma\geq\alpha}2^{d(\gamma)-k}\sum_{(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)}I^*\mu\left(\delta\right)I^*\mu\left(\delta'\right)}\\ &\leq C_n\sup_{\mathcal{T}_n}\left\|\mu\right\|_{Carleson(\mathcal{T}_n)}, \end{aligned}$$

where the supremum is taken over all $\alpha\in\mathcal{T}_n$ and $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.

We note that Theorems 21 and 36 and Lemma 31 are independent of dimension, but the argument given above to establish the equivalence of (73) with (107) and (92) does depend on dimension.

Combining the three propositions above, we obtain the following characterization of Carleson measures for the Drury-Arveson space.

Theorem 34 *A positive measure μ on the ball $\mathbb{B}_n$ is H_n^2-Carleson if and only if μ satisfies the simple condition (92) and the split tree condition (107) taken over all unitary rotations of a fixed Bergman tree. Moreover, we have*

$$\begin{aligned} c_n\left\|\mu\right\|_{Carleson} &\leq \sup_{\alpha\in\mathcal{T}_n}\sqrt{2^{d(\alpha)}I^*\mu\left(\alpha\right)}\\ &\quad+\sup_{\substack{\alpha\in\mathcal{T}_n\\ I^*\mu(\alpha)>0}}\sqrt{\frac{1}{I^*\mu\left(\alpha\right)}\sum_{k\geq 0}\sum_{\gamma\geq\alpha}2^{d(\gamma)-k}\sum_{(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)}I^*\mu\left(\delta\right)I^*\mu\left(\delta'\right)}\\ &\leq C_n\left\|\mu\right\|_{Carleson}, \end{aligned}$$

where the supremum is also taken over all unitary rotations $\mathcal{T}_n$ of a fixed Bergman tree.

Remark 35 *We can recast the above characterization on the ball as follows. For $w \in \mathbb{B}_n$ let $T(w)$ be the Carleson tent associated to w,*

$$T(w) = \{z \in \mathbb{B}_n : |1 - \overline{z} \cdot Pw| \leq 1 - |w|\},$$

and Pw denotes radial projection of w onto the sphere $\partial\mathbb{B}_n$. The H^2_n-Carleson norm of a positive measure μ on $\mathbb{B}_n$ satisfies

$$\begin{aligned} c_n \|\mu\|_{Carleson} &\leq \sup_{w \in \mathbb{B}_n} \sqrt{\left(1 - |w|^2\right)^{-1} \mu(T(w))} \\ &+ \sup_{\substack{w \in \mathbb{B}_n \\ \mu(T(w))>0}} \sqrt{\frac{1}{\mu(T(w))} \int_{T(w)} \left| \int_{T(w)} \left(\operatorname{Re} \frac{1}{1 - \overline{z} \cdot z'} \right) d\mu(z') \right|^2 d\mu(z)} \\ &\leq C_n \|\mu\|_{Carleson}. \end{aligned}$$

The comparability constants c_n and C_n in Theorem 34 depend on the dimension n because of Propositions 29 and 33, which both use an averaging process over all unitary rotations of a fixed Bergman tree. Indeed, Proposition 29 uses the lower bound (88), for a fixed proportion of rotations, for the *denominator* of the real part of the reproducing kernel in (87), while Proposition 33 uses the lower bound in (78), for a *different* fixed proportion of rotations, for the *numerator* in (87). The subsequent averaging is essentially equivalent to a covering lemma whose comparability constants depend on dimension. On the other hand Proposition 27 gives a characterization that is independent of dimension. It would be of interest, especially of view of Theorem 6, to find a more geometric characterization which is independent of dimension. In particular, we do not know if the constants in the geometric characterization in the previous remark can be taken to be independent of dimension.

4.3 Related inequalities

Inequality (77) implies

$$d(\alpha \wedge \alpha') \leq d^*([\alpha] \wedge [\alpha']) \leq \min\{d(\alpha), d(\alpha')\},$$

which has the following interpretation relative to the kernel

$$K(\alpha, \alpha') = 2^{2d(\alpha \wedge \alpha') - d^*([\alpha] \wedge [\alpha'])}$$

of the operator T_μ in (74).

If we replace $d^*([\alpha] \wedge [\alpha'])$ by the lower bound $d(\alpha \wedge \alpha')$ in the kernel $K(\alpha, \alpha')$, then T_μ becomes

$$T_\mu g(\alpha) = \sum_{\alpha' \in \mathcal{T}_n} 2^{d(\alpha \wedge \alpha')} g(\alpha') \mu(\alpha'), \tag{108}$$

whose boundedness on $\ell^2(\mu)$ is equivalent to μ being a Carleson measure for $B_2^{1/2}(\mathcal{T}_n)$, which is in turn equivalent to the tree condition (5). (Alternatively,

the above kernel is the discretization of the continuous kernel $\left|\frac{1}{1-\overline{z}\cdot z'}\right|$, whose Carleson measures are characterized by the tree condition). This observation is at the heart of Proposition 9 given earlier that shows the tree condition characterizes Carleson measures supported on a 2-manifold that meets the boundary transversely and in the complex directions (so that $d^*\left([\alpha]\wedge[\alpha']\right)\approx d\left(\alpha\wedge\alpha'\right)$ for α,α' in the support of the measure). In addition, we can see from this observation that the simple condition (92) is not sufficient for μ to be a $B_2^{1/2}\left(\mathcal{T}_n\right)$-Carleson measure. Indeed, let $\mathcal{Y}$ be any dyadic subtree of $\mathcal{T}_n$ with the properties that the two children α_+ and α_- of each $\alpha\in\mathcal{Y}$ are also children of α in $\mathcal{T}_n$, and such that no two tree elements in $\mathcal{Y}$ are equivalent. Now let μ be any measure supported on $\mathcal{Y}$ that satisfies the simple condition

$$2^{d(\alpha)}I^*\mu\left(\alpha\right)\leq C,\qquad \alpha\in\mathcal{Y},$$

but not the tree condition

$$\sum_{\beta\in\mathcal{Y}:\beta\geq\alpha}\left[2^{d(\beta)/2}I^*\mu\left(\beta\right)\right]^2\leq CI^*\mu\left(\alpha\right)<\infty,\qquad \alpha\in\mathcal{Y}.$$

For $\alpha,\alpha'\in\mathcal{Y}$, we have $d\left([\alpha]\wedge[\alpha']\right)=d\left(\alpha\wedge\alpha'\right)$, and so μ is a $B_2^{1/2}\left(\mathcal{T}_n\right)$-Carleson measure if and only if the operator T in (108) is bounded on $\ell^2\left(\mu\right)$, which is equivalent to the above tree condition, which we have chosen to fail. Finally, to transplant this example to the ball $\mathbb{B}_n$, we take $d\mu\left(z\right)=\sum_{\alpha\in\mathcal{Y}}\mu\left(\alpha\right)\delta_{c_\alpha}\left(z\right)$ and show that the above tree condition fails on a positive proportion of the rotated trees $U^{-1}\mathcal{T}_n$, $U\in\mathcal{U}_n$.

If on the other hand, we replace $d^*\left([\alpha]\wedge[\alpha']\right)$ in the kernel $K\left(\alpha,\alpha'\right)$ by the upper bound $\min\left\{d\left(\alpha\right),d\left(\alpha'\right)\right\}$, then T_μ becomes

$$T_\mu g\left(\alpha\right)=\sum_{\alpha'\in\mathcal{T}_n}2^{2d\left(\alpha\wedge\alpha'\right)-\min\left\{d(\alpha),d\left(\alpha'\right)\right\}}g\left(\alpha'\right)\mu\left(\alpha'\right),\tag{109}$$

whose boundedness on $\ell^2\left(\mu\right)$ is shown in Theorem 36 below to be implied by the simple condition (92). Thus we see that the simple condition (92) characterizes Carleson measures supported on a slice (when $d^*\left([\alpha]\wedge[\alpha']\right)=\min\left\{d\left(\alpha\right),d\left(\alpha'\right)\right\}$ for α,α' in the support of the measure). In particular, this provides a new proof that the simple condition (92) characterizes Carleson measures for the Hardy space $H^2\left(\mathbb{B}_1\right)=B_2^{1/2}\left(\mathbb{B}_1\right)$ in the unit disc. A more general result based on this type of estimate was given in Proposition 7.

For the sake of completeness, we note that the above inequalities (108) and (109) correspond to the two extreme estimates in (77) for the *second* terms on the right sides of (80) and (81). The *first* term c on the right side of (80) leads to the operator

$$T_\mu g\left(\alpha\right)=\sum_{\alpha'\in\mathcal{T}_n}g\left(\alpha'\right)\mu\left(\alpha'\right),$$

whose boundedness on $\ell^2\left(\mu\right)$ is trivially characterized by finiteness of the measure μ.

As a final instance of the split tree condition simplifying when there is additional geometric information we consider measures which are invariant under the natural action of the circle on the ball. Here we extend the language of [5] where measures on spheres were considered and say a measure ν on $\mathbb{B}_n$ is *invariant* if

$$\int_{\mathbb{B}_n} f\left(e^{i\theta_1}z_1,...,e^{i\theta_n}z_n\right)d\nu\left(z\right)=\int_{\mathbb{B}_n} f\left(z\right)d\nu\left(z\right)$$

for all continuous functions f on the ball. We will also abuse the terminology and use it for the discretization of such a measure.

We want to know when there is a Carleson embedding for such a measure. In fact, when μ is invariant, the operator T_μ in (74) is bounded on $\ell^2\left(\mu\right)$ if and only if μ is finite. To see this we need the "Poisson kernel" estimate

$$\sum_{\beta\in B}2^{2d(\alpha\wedge\beta)}\approx 2^{d(B)+d([\alpha]\wedge B)},\qquad \alpha\in\mathcal{T}_n, B\in\mathcal{R}_n. \tag{110}$$

With $A=[\alpha]$ and $\alpha^*=E_{A\wedge B}\alpha$, (110) follows from

$$\begin{aligned}\sum_{\beta\in B}2^{2d(\alpha\wedge\beta)} &= \sum_{\gamma\in A\wedge B}\sum_{\substack{\beta\in B\\ \beta\geq\gamma}}2^{2d(\alpha^*\wedge\gamma)}\\ &= 2^{d(B)-d(A\wedge B)}\sum_{j=0}^{d(A\wedge B)}2^{2j}2^{d(A\wedge B)-j}\\ &\approx 2^{d(B)-d(A\wedge B)}2^{2d(A\wedge B)}.\end{aligned}$$

Now with $\mu\left(A\right)=\sum_{\alpha\in A}\mu\left(\alpha\right)$, and recalling that μ is invariant, we have for $\alpha\in A$,

$$\begin{aligned}T_\mu f\left(\alpha\right) &\leq \sum_{B\in\mathcal{R}_n}\sum_{\beta\in B}2^{2d(\alpha\wedge\beta)-d(A\wedge B)}f\left(\beta\right)\mu\left(\beta\right)\\ &\approx \sum_{B\in\mathcal{R}_n}\mu\left(B\right)2^{-d(A\wedge B)-d(B)}\sum_{\beta\in B}2^{2d(\alpha\wedge\beta)}f\left(\beta\right).\end{aligned}$$

Using (110) we compute that $T_\mu 1$ is bounded (and hence a Schur function):

$$T_\mu 1\left(\alpha\right)\approx\sum_{B\in\mathcal{R}_n}\mu\left(B\right)2^{-d(A\wedge B)-d(B)}\sum_{\beta\in B}2^{2d(\alpha\wedge\beta)}\approx\sum_{B\in\mathcal{R}_n}\mu\left(B\right)=\left\|\mu\right\|.$$

Thus T_μ is bounded on $\ell^\infty\left(\mu\right)$ with norm at most $\left\|\mu\right\|$, and by duality also on $\ell^1\left(\mu\right)$. Interpolation now yields that T_μ is bounded on $\ell^2\left(\mu\right)$ with norm at most $\left\|\mu\right\|$.

Theorem 36 *Let $0 < r < \infty$. A positive measure μ satisfies the bilinear inequality*

$$\sum_{\alpha,\alpha' \in \mathcal{T}_n} 2^{(1+r)d\left(\alpha \wedge \alpha'\right) - r\min\left\{d(\alpha), d\left(\alpha'\right)\right\}} f(\alpha)\,\mu(\alpha)\,g\left(\alpha'\right)\mu\left(\alpha'\right) \tag{111}$$

$$\leq C\,\|f\|_{\ell^2(\mathcal{T}_n;\mu)}\,\|g\|_{\ell^2(\mathcal{T}_n;\mu)}, \tag{112}$$

if μ satisfies the simple condition (92). Moreover, the constant implicit in this statement is independent of n.

Remark 37 *The proof below shows that the ratio of the constant C in (111) to that in (92) is $O\left(\frac{1}{r}\right)$. The theorem actually fails if $r = 0$. Indeed, (111) is then equivalent to the boundedness of (108) on $\ell^2(\mu)$, which as we noted above is equivalent to the tree condition (61) with $\sigma = 1/2$ and $p = 2$.*

Remark 38 *Using the argument on pages 538-542 of [33], it can be shown that the case $r = 1$ of the bilinear inequality (111) holds if and only if the following pair of dual conditions hold:*

$$\sum_{\beta \geq \alpha} \left|\mathcal{I}2^d\left(\chi_{S(\alpha)}\mu\right)(\beta)\right|^2 \mu(\beta) \leq C \sum_{\beta \geq \alpha} \mu(\beta) < \infty, \qquad \alpha \in \mathcal{T}_n, \tag{113}$$

$$\sum_{\beta \geq \alpha} \left|2^d\mathcal{I}\left(\chi_{S(\alpha)}2^{-d}\mu\right)(\beta)\right|^2 \mu(\beta) \leq C \sum_{\beta \geq \alpha} 2^{-2d(\beta)}\mu(\beta), \qquad \alpha \in \mathcal{T}_n,$$

where $\mathcal{I}$ is the fractional integral of order one on the Bergman tree given by,

$$\mathcal{I}\nu(\alpha) = \sum_{\beta \in \mathcal{T}_n} 2^{-d(\alpha,\beta)}\nu(\beta), \qquad \alpha \in \mathcal{T}_n. \tag{114}$$

We leave the lengthy but straightforward details to the interested reader. One can also use the argument given below, involving segments of geodesics, to show that the simple condition implies both conditions in (113).

We shall use the following simple sufficient condition of Schur type for the proof of Theorem 36. Recall that a measure space (Z, μ) is σ-finite if $Z = \cup_{N=1}^{\infty} Z_N$ where $\mu(Z_N) < \infty$, and that a function k on $Z \times Z$ is σ-bounded if $Z = \cup_{N=1}^{\infty} Z_N$ where k is bounded on $Z_N \times Z_N$.

Lemma 39 *(Vinogradov-Seničkin Test, pg. 151 of [29]) Let (Z, μ) be a σ-finite measure space and k a nonnegative σ-bounded function on $Z \times Z$ satisfying*

$$\int\int_{Z\times Z} k(s,t)\,k(s,x)\,d\mu(s) \leq M\left(\frac{k(t,x) + k(x,t)}{2}\right) \text{ for } \mu\text{-a.e. } (t,x) \in Z\times Z. \tag{115}$$

Then the linear map T defined by

$$Tg(s) = \int_Z k(s,t)\,g(t)\,d\mu(t)$$

is bounded on $L^2(\mu)$ with norm at most M.

Proof. Let $Z = \cup_{N=1}^{\infty} Z_N$ where $\mu(Z_N) < \infty$ and k is bounded on $Z_N \times Z_N$. The kernels

$$k_N(s,t) = k(s,t)\,\chi_{Z_N \times Z_N}(s,t)$$

satisfy (115) uniformly in N, and the corresponding operators

$$T_N g(s) = \int_Z k_N(s,t)\, g(t)\, d\mu(t)$$

are bounded on $L^2(\mu)$ (with norms depending on $\mu(Z_N)$ and the bound for k on $Z_N \times Z_N$). However, (115) for k_N implies that the integral kernel of the operator $T_N^* T_N$ is dominated pointwise by $\frac{M}{2}$ times that of $T_N^* + T_N$, and this gives $\|T_N\|^2 = \|T_N^* T_N\| \leq \frac{M}{2}\|T_N^* + T_N\| \leq M\|T_N\|$, and hence $\|T_N\| \leq M$. Now let $N \to \infty$ and use the monotone convergence theorem to obtain $\|T\| \leq M$.

Remark 40 *If $k(x,y) = k(y,x)$ is symmetric, then (115) ensures that for any choice of a, $k(a,\cdot)$ can be used as a test function for Schur's Lemma.*

Proof. (of Theorem 36) We will show that (111) holds but we first note that it suffices to consider a modified bilinear form. Set

$$k(\alpha,\beta) = 2^{(1+r)d(\alpha\wedge\beta) - rd(\beta)} \chi_{\{d(\alpha)\geq d(\beta)\}}.$$

We will consider (cf. page 152 of [29])

$$B(f,g) = \sum_{\alpha,\beta\in\mathcal{T}_n} k(\alpha,\beta) f(\alpha)\,\mu(\alpha)\, g(\beta)\,\mu(\beta)$$

because (modulo double bookkeeping on the diagonal) the form of interest is $B(f,g) + B(g,f)$. The result will follow from the lemma if we show that

$$\sum_{\alpha\in\mathcal{T}_n} k(\alpha,\beta)k(\alpha,\gamma)\mu(\alpha) \leq c(k(\beta,\gamma) + k(\gamma,\beta)).$$

The sum on the right dominates our original kernel. Thus it suffices to show that

$$\sum_{\alpha} 2^{(1+r)d(\alpha\wedge\beta) - rd(\beta)} \chi_{\{d(\alpha)\geq d(\beta)\}} 2^{(1+r)d(\alpha\wedge\gamma) - rd(\gamma)} \chi_{\{d(\alpha)\geq d(\gamma)\}} \mu(\alpha)$$
$$\leq c 2^{(1+r)d(\beta\wedge\gamma) - r\min\{d(\beta),d(\gamma)\}}.$$

Select $\beta,\gamma \in \mathcal{T}_n$. Without loss of generality we assume $d(\beta) \leq d(\gamma)$. We consider three segments of geodesics in $\mathcal{T}_n$; Γ_1 connecting β to $\beta\wedge\gamma$, Γ_2 connecting γ to $\beta\wedge\gamma$, and Γ_3 connecting $\beta\wedge\gamma$ to the root o. Denote the lengths of the three by k_i, $i = 1,2,3$. (It is not a problem if some segments are degenerate.) Set $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. We consider three subsums, those where the geodesic from α to o first encounters Γ at a point in Γ_i, $i = 1,2,3$.

We first consider the case $i = 1$. Let $\delta_{k_3}, \delta_{k_3+1}$,...,$\delta_{k_3+k_1}$ be an enumeration of the points of Γ_1 starting at $\beta \wedge \gamma$; thus $d(\delta_j) = j$. For $\alpha \in S(\delta_j)$ we have $\alpha \wedge \beta = \delta_j$ and $\alpha \wedge \gamma = \beta \wedge \gamma = \delta_{k_3}$. Thus

$$\begin{aligned}
\sum(1,j) &= \sum_{\alpha \in S(\delta_j)} 2^{(1+r)d(\alpha\wedge\beta)-rd(\beta)}\chi_{\{d(\alpha)\geq d(\beta)\}}2^{(1+r)d(\alpha\wedge\gamma)-rd(\gamma)}\chi_{\{d(\alpha)\geq d(\gamma)\}}\mu(\alpha) \\
&\leq \sum_{\alpha \in S(\delta_j)} 2^{(1+r)j-rd(\beta)}2^{(1+r)2k_3-rd(\gamma)}\mu(\alpha) \\
&= \sum_{\alpha \in S(\delta_j)} 2^{(1+r)j-r(k_3+k_1)+(1+r)k_3-r(k_3+k_2)}\mu(\alpha) \\
&\leq c2^{(1+r)j-r(k_3+k_1)+(1+r)k_3-r(k_3+k_2)-j} \\
&= c2^{rj-rk_1-rk_2+(1-r)k_3}
\end{aligned}$$

where the second inequality uses the simple condition on μ. Summing these estimates gives

$$\begin{aligned}
\sum_{j=k_3}^{k_3+k_1}\sum(1,j) &\leq c2^{r(k_3+k_1)-rk_1-rk_2+(1-r)k_3} \\
&= c2^{k_3-rk_2} \\
&\leq c2^{(1+r)k_3-r\min\{k_3+k_1,k_3+k_2\}} \\
&= c2^{(1+r)d(\beta\wedge\gamma)-r\min\{d(\beta),d(\gamma)\}}
\end{aligned}$$

as required.

The other two cases are similar. Let $\tau_{k_3}, ..., \tau_{k_3+k_2}$ be a listing of the points of Γ_2 starting at the top. Then

$$\begin{aligned}
\sum(2,j) &= \sum_{\alpha \in S(\tau_j)} 2^{(1+r)d(\alpha\wedge\beta)-rd(\beta)}\chi_{\{d(\alpha)\geq d(\beta)\}}2^{(1+r)d(\alpha\wedge\gamma)-rd(\gamma)}\chi_{\{d(\alpha)\geq d(\gamma)\}}\mu(\alpha) \\
&\leq \sum_{\alpha \in S(\tau_j)} 2^{(1+r)k_3-rd(\beta)}2^{(1+r)j-rd(\gamma)}\mu(\alpha) \\
&\leq c2^{(1+r)j-r(k_3+k_1)+(1+r)k_3-r(k_3+k_2)-j}.
\end{aligned}$$

Hence

$$\begin{aligned}
\sum_{j=k_3}^{k_3+k_2}\sum(2,j) &\leq c2^{r(k_3+k_2)-rk_1-rk_2+(1-r)k_3} \\
&= c2^{k_3-rk_1} \\
&\leq c2^{(1+r)k_3-r\min\{k_3+k_1,k_3+k_2\}} \\
&= c2^{(1+r)d(\beta\wedge\gamma)-r\min\{d(\beta),d(\gamma)\}}
\end{aligned}$$

as required.

In the final case, let $\rho_0, ..., \rho_{k_3}$ be a listing of the points of Γ_3 starting at the top. Then

$$\begin{aligned}\sum(3,j) &= \sum_{\alpha\in S(\rho_j)} 2^{(1+r)d(\alpha\wedge\beta)-rd(\beta)}\chi_{\{d(\alpha)\geq d(\beta)\}}2^{(1+r)d(\alpha\wedge\gamma)-rd(\gamma)}\chi_{\{d(\alpha)\geq d(\gamma)\}}\mu(\alpha)\\ &= \sum_{\alpha\in S(\rho_j)} 2^{(1+r)j-rd(\beta)}\chi_{\{d(\alpha)\geq d(\beta)\}}2^{(1+r)j-rd(\gamma)}\chi_{\{d(\alpha)\geq d(\gamma)\}}\mu(\alpha)\\ &\leq \sum_{\alpha\in S(\rho_j)} 2^{(1+r)j-rd(\beta)}2^{(1+r)j-rd(\gamma)}\mu(\alpha)\\ &\leq c2^{(1+r)j-r(k_3+k_1)+(1+r)j-r(k_3+k_2)-j} = c2^{(1+2r)j-r(k_3+k_1)-r(k_3+k_2)},\end{aligned}$$

and hence

$$\begin{aligned}\sum_{j=0}^{k_3}\sum(3,j) &\leq c2^{(1+2r)k_3-r(k_1+k_2)-2rk_3}\\ &= c2^{k_3-r(k_1+k_2)}\\ &\leq c2^{(1+r)k_3-r\min\{k_3+k_1,k_3+k_2\}}\\ &= c2^{(1+r)d(\beta\wedge\gamma)-r\min\{d(\beta),d(\gamma)\}}\end{aligned}$$

and we are done.

5 Appendix: Nonisotropic potential spaces

Define the nonisotropic potential spaces $\mathcal{P}_\alpha^2(\mathbb{B}_n)$, $0<\alpha<n$, to consist of all potentials $K_\alpha f$ of L^2 functions on the sphere $\mathbb{S}_n = \partial\mathbb{B}_n$, $f\in L^2(d\sigma_n)$, where

$$K_\alpha f(z) = \int_{\mathbb{S}_n} \frac{f(\zeta)}{\left|1-\overline{\zeta}\cdot z\right|^{n-\alpha}} d\sigma_n(\zeta), \qquad z\in\mathbb{B}_n.$$

Thus, with $\alpha = 2\gamma$, these spaces are closely related to the spaces of holomorphic functions $J_\gamma^2(\mathbb{B}_n)$ defined in the introduction. It is pointed out in [18] that Carleson measures μ for the potential space $\mathcal{P}_\alpha^2(\mathbb{B}_n)$, i.e. those measures μ satisfying

$$\int_{\mathbb{B}_n} |K_\alpha f(z)|^2 d\mu(z) \leq C\int_{\mathbb{S}_n} |f(\zeta)|^2 d\sigma_n(\zeta), \tag{116}$$

can be characterized by a capacitary condition involving a nonisotropic capacity $C_\alpha(A)$ and nonisotropic tents $T(A)$ defined for open subsets A of $\mathbb{S}_n$:

$$\mu(T(A)) \leq CC_\alpha(A), \qquad \text{for all } A \text{ open in } \mathbb{S}_n. \tag{117}$$

The dual of the Carleson measure inequality for the nonisotropic potential space $\mathcal{P}^2_{\frac{n}{2}-\sigma}(\mathbb{B}_n)$ is

$$\left\|T_\mu^\sigma g\right\|_{L^2(\sigma_n)} \leq C\|g\|_{L^2(\mu)}, \qquad g\in L^2(\mu), \tag{118}$$

where the operator T_{μ}^{σ} is given by

$$T_{\mu}^{\sigma} g(w) = \int_{\mathbb{S}_n} \frac{1}{|1-\overline{z}\cdot w|^{\frac{n}{2}+\sigma}} g(z)\, d\mu(z).$$

The Carleson measure inequality for $B_2^{\sigma}(\mathbb{B}_n)$ is equivalent to

$$\left\|S_{\mu}^{\sigma} g\right\|_{L^2(\lambda_n)} \leq C \|g\|_{L^2(\mu)}, \qquad g \in L^2(\mu),$$

where the operator S_{μ}^{σ} is given by

$$S_{\mu}^{\sigma} g(w) = \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{-\sigma} \left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{2}+\sigma} g(z)\, d\mu(z),$$

for any choice of $\alpha > -1$. It is easy to see that the tree condition (3) characterizes the inequality

$$\left\|T_{\mu}^{\sigma,\alpha} g\right\|_{L^2(\lambda_n)} \leq C_{\alpha} \|g\|_{L^2(\mu)}, \qquad g \in L^2(\mu), \tag{119}$$

where the operator $T_{\mu}^{\sigma,\alpha}$ is given by

$$T_{\mu}^{\sigma,\alpha} g(w) = \int_{\mathbb{B}_n} \frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{2}}}{|1-\overline{z}\cdot w|^{\frac{n+1+\alpha}{2}+\sigma}} g(z)\, d\mu(z).$$

Moreover, the constants C_{α} in (119) and C in (3) satisfy

$$C_{\alpha}^2 \approx (1+\alpha)^{-1} C, \qquad \alpha > -1. \tag{120}$$

Now if we use (120) to rewrite (119) for $g \geq 0$ as

$$\begin{aligned} &\int_{\mathbb{B}_n} \left\{ \int_{\mathbb{B}_n} \frac{1}{|1-\overline{z}\cdot w|^{\frac{n+1+\alpha}{2}+\sigma}} g(z)\, d\mu(z) \right\}^2 (1+\alpha)\left(1-|w|^2\right)^{\alpha} dw \qquad (121) \\ &\quad \leq C \int_{\mathbb{B}_n} g(z)^2\, d\mu(z), \end{aligned}$$

we obtain the following result.

Theorem 41 *Inequality (121) holds for* some $\alpha > -1$ *if and only if (121) holds for* all $\alpha > -1$ *if and only if (118) holds if and only if the tree condition (61) holds.*

Proof. If (121) holds for some $\alpha > -1$, then the tree condition (61) holds. If the tree condition (61) holds, then (121) holds for all $\alpha > -1$ with a constant C independent of α. If we let $\alpha \to -1$ and note that

$$(1+\alpha)\left(1-|w|^2\right)^{\alpha} dw \to c_n d\sigma_n, \qquad as\ \alpha \to -1,$$

we obtain that (118) holds. Finally, if (118) holds, then it also holds with $T_{\mu}^{\sigma} g(rw)$ in place of $T_{\mu}^{\sigma} g(w)$ for $g \geq 0$ and all $0 < r < 1$, and an appropriate integration in r now yields (121) for all $\alpha > -1$.

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