BILINEAR FORMS ON THE DIRICHLET SPACE

1. Introduction

For $1 < r < \infty$, let $dA_r(z) := \left(1 - |z|^2\right)^{r-2} dA(z)$. The Dirichlet space $\mathcal{D}_r$ is the collection of functions that are analytic on the unit disc $\mathbb{D}$ such that the following norm is finite,

$$\|f\|_{\mathcal{D}_r}^r := |f(0)|^r + \int_{\mathbb{D}} |f'(z)|^r \, dA_r(z)$$

For $s > -1$ we define two different linear operators that will act on the space $L^p(\mathbb{D}; dA_p)$. We first have

$$\mathbb{P}_s(f)(z) := c_s \int_{\mathbb{D}} \frac{\left(1 - |w|^2\right)^s}{(1 - z\overline{w})^{2+s}} f(w) dA(w).$$

We will also need a variant of this operator, but where we taken the absolute value of the kernel. We set

$$\mathfrak{P}_s(f)(z) := c_s \int_{\mathbb{D}} \frac{\left(1 - |w|^2\right)^s}{|1 - z\overline{w}|^{2+s}} f(w) dA(w).$$

It is well known that for $s > -1$ these operators are bounded on $L^p(\mathbb{D}; dA_p)$.

Now define a bilinear form T_b on the space of polynomials $\mathcal{P}$ on the disk by

$$T_b(f, g) \equiv \langle fg, b \rangle_{\mathcal{D}_2}, \qquad f, g \in \mathcal{P},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{D}_2}$ is the inner product for the Dirichlet space $\mathcal{D} = \mathcal{D}_2$ given by

$$\langle f, g \rangle_{\mathcal{D}} = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)} dA(z).$$

We can no longer assert that the norm $\|H_b\|_{\mathcal{D}}$ of the Hankel operator H_b from $\mathcal{D}$ to $\mathcal{D}_-$ is the same as the norm $\|T_b\|_{\mathcal{D}}$ of the bilinear form T_b on $\mathcal{D}_p \times \mathcal{D}_q$, since the inner product for the Dirichlet space involves derivatives. For a positive measure μ on the disk, let $\|\mu\|_{\mathcal{D}-Carleson}$ be the (possibly infinite) norm of the inclusion $\mathcal{P} \subset L^2(\mu)$ with inner product $\langle \cdot, \cdot \rangle_{\mathcal{D}}$ on $\mathcal{P}$. It is shown in Rochberg and Wu [6] that $\|H_b\|_{\mathcal{D}} \approx |b(0)| + \|\mu_b\|_{\mathcal{D}-Carleson}$. Here we show the same for T_b.

Theorem 1. *Let b be holomorphic on the unit disc $\mathbb{D}$. Then T_b extends to a bounded bilinear form on $\mathcal{D}_p \times \mathcal{D}_q$ if and only if for $r = p, q$ the measure $d\mu_{b,r}(z) \equiv |b'(z)|^r dA_r(z)$ is a Carleson measure for the Dirichlet space $\mathcal{D}_r$. Moreover,*

$$\|T_b\| \approx |b(0)| + \|\mu_{b,p}\|_{\mathcal{D}_p-Carleson} + \|\mu_{b,q}\|_{\mathcal{D}_q-Carleson}.$$

2. Proof of the theorem

Suppose first that for $r = p, q$, $\mu_{b,r}$ is a $\mathcal{D}_r$-Carleson measure. For $f, g \in \mathcal{P}$ we have

$$\begin{aligned}
\left|T_b(f,g)\right| &= \left| f(0)\,g(0)\,\overline{b(0)} + \int_{\mathbb{D}} \left(f'(z)\,g(z) + f(z)\,g'(z)\right)\overline{b'(z)}dA(z)\right| \\
&\leq \left|f(0)\,g(0)\,b(0)\right| + \int_{\mathbb{D}} \left|f'(z)\,g(z)\,b'(z)\right| dA(z) + \int_{\mathbb{D}} \left|f(z)\,g'(z)\,b'(z)\right| dA(z) \\
&\leq \left|f(0)\,g(0)\,b(0)\right| + \left(\int_{\mathbb{D}} \left|f'(z)\right|^p dA_p(z)\right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} \left|g(z)\right|^q d\mu_{b,q}(z)\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{\mathbb{D}} \left|g'(z)\right|^q dA_q(z)\right)^{\frac{1}{q}} \left(\int_{\mathbb{D}} \left|f(z)\right|^p d\mu_{b,p}(z)\right)^{\frac{1}{p}} \\
&\leq C\left(\left|b(0)\right| + \left\|\mu_{b,p}\right\|_{\mathcal{D}_p-Carleson} + \left\|\mu_{b,q}\right\|_{\mathcal{D}_q-Carleson}\right)\left\|f\right\|_{\mathcal{D}_p}\left\|g\right\|_{\mathcal{D}_q}.
\end{aligned}$$

Thus T_b has a bounded extension to $\mathcal{D}_p \times \mathcal{D}_q$ with

$$\left\|T_b\right\| \leq C\left(\left|b(0)\right| + \left\|\mu_{b,p}\right\|_{\mathcal{D}_p-Carleson} + \left\|\mu_{b,q}\right\|_{\mathcal{D}_q-Carleson}\right).$$

Conversely, suppose that T_b extends to a bounded bilinear form on $\mathcal{D}_p \times \mathcal{D}_q$. Then with $g = 1$ we obtain

$$\left|\langle f, b\rangle_{\mathcal{D}}\right| = \left|T_b(f,1)\right| \leq \left\|T_b\right\|\left\|f\right\|_{\mathcal{D}_p}\left\|1\right\|_{\mathcal{D}_q}$$

for all polynomials $f \in \mathcal{P}$, which shows that $b \in \mathcal{D}_q$ and

$$\left\|b\right\|_{\mathcal{D}_q} \leq C\left\|T_b\right\|. \tag{2.1}$$

Repeating this argument, but interchanging the roles of p and q, we also see that,

$$\left\|b\right\|_{\mathcal{D}_p} \leq C\left\|T_b\right\|. \tag{2.2}$$

Also, note that letting $f = g = 1$ we see that

$$|b(0)| \leq \left\|T_b\right\|. \tag{2.3}$$

We next observe that is suffices to prove only one of the measures is Carleson for the appropriate space. Suppose that we have shown $\|\mu_{b,p}\|_{\mathcal{D}_p-Carleson} \lesssim \|T_b\|$. Then, it is easy to see that the bilinear form $F_b : \mathcal{D}_p \times \mathcal{D}_q \to \mathbb{C}$ given by

$$F_b(f,g) := T_b(f,g) - \int_{\mathbb{D}} \overline{b'(z)} f(z) g'(z) dA(z) - \overline{b(0)} f(0) g(0) = \int_{\mathbb{D}} \overline{b'(z)} f'(z) g(z) dA(z)$$

is also bounded with norm controlled by

$$\begin{aligned}
\left\|F_b\right\| &\leq 2\left\|T_b\right\| + \left\|\mu_{b,p}\right\|_{\mathcal{D}_p-Carleson} \\
&\lesssim \left\|T_b\right\|
\end{aligned}$$

with the last line following from the supposition that we already knew the estimate for the norm of the $\mathcal{D}_p$-Carleson measure $\mu_{b,p}$ was controlled by $\|T_b\|$. But, it is also easy to see that

$$\left\|F_b\right\| \approx \left\|\mu_{b,q}\right\|_{\mathcal{D}_q-Carleson}$$

and so we can conclude that $\|\mu_{b,q}\|_{\mathcal{D}_q-Carleson} \lesssim \|T_b\|$. Thus, it suffices to show that *one* of the measures $\mu_{b,q}$ or $\mu_{b,p}$ is Carleson for the appropriate space with

Carleson measure controlled by $\|T_b\|$, the other follows from the above argument. Additionally, because of this, we can suppose that $p < 2 < q$, and we only need to show that $\mu_{b,q}$ is $\mathcal{D}_q$-Carleson.

2.1. **Sketch of Proof.** Let $\{I_j\}$ be a finite collection of disjoint intervals in $\mathbb{T}$ and let $\cup_j T(I_j)$ denote the Carleson tents in $\mathbb{D}$. We will chose the collection of intervals $\{I_j\}$ later to extremize a capacity problem.

Set $\beta_q(z) := |b'(z)|^{q-2}\left(1-|z|^2\right)^{q-2}$. Define the following function

$$f_q(z) := \int_{\mathbb{D}} \frac{\left(1-|\xi|^2\right)^s}{\bar{\xi}\left(1-\bar{\xi}z\right)^{1+s}} b'(\xi)\beta_q(\xi)\chi_{\cup_j T(I_j)}(\xi)dA(\xi)$$

Then $f_q \in \mathcal{D}_p$ since one can show that $|f(0)| \lesssim \|T_b\|^{q-1}$ and

$$\left(\int_{\mathbb{D}} \left|f_q'(z)\right|^p dA_p(z)\right)^{1/p} \lesssim \left(\int_{\mathbb{D}} |b'(z)|^q \, dA_q(z)\right)^{1/p} \leq \|T_b\|^{q-1},$$

so $\|f_q\|_{\mathcal{D}_p} \lesssim \|T_b\|^{q-1} < \infty$. Also, observe that for $G = \cup_j T(I_j)$ and $\widetilde{G}$ denoting an "enlargement" of the set G (done in such a way that $\operatorname{cap}_q \widetilde{G} \approx \operatorname{cap}_q G$) we have

$$\begin{aligned}
f_q'(z) &= \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z) \\
&= b'(z)\beta_q(z)\chi_{G)}(z) + \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z) - b'(z)\beta_q(z)\chi_G(z) \\
&= b'(z)\beta_q(z)\chi_G(z) + \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z) - \mathbb{P}_s\left(b'\right)(z)\beta_q(z)\chi_{G)}(z) \\
&= b'(z)\beta_q(z)\chi_G(z) + \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\chi_{\widetilde{G}}(z) + \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\chi_{\widetilde{G}^c}(z) \\
&\quad -\mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\beta_q(z)\chi_G(z) - \mathbb{P}_s\left(b'\chi_{\widetilde{G}^c}\right)(z)\beta_q(z)\chi_G(z) \\
&= b'(z)\beta_q(z)\chi_G(z) + \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\chi_{\widetilde{G}^c}(z) - \mathbb{P}_s\left(b'\chi_{\widetilde{G}^c}\right)(z)\beta_q(z)\chi_G(z) \\
&\quad -\mathbb{P}_s\left(b'\beta_q\chi_{\widetilde{G}\setminus G}\right)(z)\chi_{\widetilde{G}}(z) + \mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\beta_q(z)\chi_{\widetilde{G}\setminus G}(z) + \left[\mathbb{P}_s, \beta_q\right](b'\chi_{\widetilde{G}})\chi_{\widetilde{G}}(z) \\
&:= b'(z)\beta_q(z)\chi_G(z) + E_q(b')(z).
\end{aligned}$$

Thus, $f_q'(z)$ is $b'(z)\beta_q(z)$ localized to the set $\cup_j T(I_j)$, up to an error given by a sum of commutator type terms. Adding and subtracting common terms one can see that the commutator term can be decomposed into parts localized to the set $\cup_j T(I_j)$ and its complement. Namely,

$$\begin{aligned}
E_q(b')(z) &= \mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\chi_{\widetilde{G}^c}(z) - \mathbb{P}_s\left(b'\chi_{\widetilde{G}^c}\right)(z)\beta_q(z)\chi_G(z) \\
&\quad -\mathbb{P}_s\left(b'\beta_q\chi_{\widetilde{G}\setminus G}\right)(z)\chi_{\widetilde{G}}(z) + \mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\beta_q(z)\chi_{\widetilde{G}\setminus G}(z) \\
&\quad + \left[\mathbb{P}_s, \beta_q\right](b'\chi_{\widetilde{G}})\chi_{\widetilde{G}}(z)
\end{aligned}$$

Note that when $q = 2$ the last commutator above vanishes since $\beta_2(z) \equiv 1$.

Let φ be an extremal for the capacity of the set of intervals. We use the dyadic tree on the unit disc to construct this function. We then set $g := \varphi^2$. If we substitute these functions into the bilinear form T_b, we find

$$\begin{aligned}
T_b(f_q,g) &= \overline{b(0)}f(0)g(0)+\int_{\mathbb{D}}\overline{b'(z)}\left(f_q(z)g'(z)+f_q'(z)g(z)\right)dA(z)\\
&= \overline{b(0)}f(0)g(0)+\int_{\mathbb{D}}\overline{b'(z)}b'(z)\beta_q(z)\chi_{\cup_j T(I_j)}(z)g(z)dA(z)\\
&\quad+\int_{\mathbb{D}}\overline{b'(z)}E_q(b')(z)g(z)dA(z)+\int_{\mathbb{D}}\overline{b'(z)}f_q(z)g'(z)dA(z)\\
&= (1)+(2)+(3)+(4).
\end{aligned}$$

We need to estimate each of the terms (1), (2), (3), (4), and $|T_b(f_q,g)|$. We will prove either these terms can be estimated by $\|T_b\|^q\operatorname{cap}_q(\cup_j I_j)$ or

$$\epsilon\mu_{b,q}\left(\cup_j T\left(I_j\right)\right)+C(\epsilon)\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)$$

where $\epsilon>0$ is a small number that can be chosen at the end.

With these estimates, we conclude the proof as follows. First, observe that

$$\begin{aligned}
\mu_{b,q}\left(\cup_{j=1}^N T(I_j)\right) &= (2)-C\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_j I_j\right)\\
&= T_b(f_q,g)-(1)-(3)-(4)-C\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right).
\end{aligned}$$

Then, taking absolute values and using the estimates we claim, we see that

$$\begin{aligned}
\mu_{b,q}\left(\cup_{j=1}^N T(I_j)\right) &\le |T_b(f_q,g)|+|(1)|+|(3)|+|(4)|+C\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)\\
&\le \epsilon C\mu_{b,q}\left(\cup_{j=1}^N T\left(I_j\right)\right)+C(\epsilon)\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right).
\end{aligned}$$

Choosing ϵ sufficiently small, we see

$$\mu_{b,q}\left(\cup_{j=1}^N T(I_j)\right)\lesssim\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right),$$

and so $\mu_{b,q}$ is a $\mathcal{D}_q$-Carleson measure. This would then prove the Theorem.

2.2. **Term** (1)**:** Notice that term (1) is trivial. We have that $|b(0)|\le\|T_b\|$, $|f(0)|\lesssim\|T_b\|^{q-1}$ and $|g(0)|\lesssim\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)$, so

$$|(1)|\lesssim\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right).$$

2.3. **Term** (2)**:** Next, note that term (2) is also easy to handle. By the definition of $\beta_q(z)$ we have

$$(2)=\int_{\mathbb{D}}\left|b'(z)\right|^q\left(1-|z|^2\right)^{q-2}\chi_{\cup_j T(I_j)}(z)g(z)dA(z)$$

But, by construction we have that $g(z)=1+C\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)$ on the set $\cup_j T(I_j)$, and so we have

$$\begin{aligned}
(2) &= \mu_{b,q}\left(\cup_j T(I_j)\right)+C\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)\int_{\mathbb{D}}\left|b'(z)\right|^q dA_q(z)\\
&= \mu_{b,q}\left(\cup_j T(I_j)\right)+C\left\|b\right\|_{\mathcal{D}_q}^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)\\
&= \mu_{b,q}\left(\cup_j T(I_j)\right)+O\left(\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_{j=1}^N I_j\right)\right),
\end{aligned}$$

which is the estimate that we seek.

2.4. **Term** (3)**:** Recall that we are letting $G=\cup_{j=1}^N I_j$ and $\widetilde{G}$ is denoting an enlargement of the set G done in such a way so that the $\text{cap}_q(\widetilde{G})\approx\text{cap}_q(G)$. Using the decomposition of $E_q(b')$ we see that term (3) decomposes as

$$\begin{aligned}
(3) &= \int_{\mathbb{D}}\overline{b'(z)}E_q(b')(z)g(z)dA(z)\\
&= \int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\chi_{\widetilde{G}^c}(z)g(z)dA(z)\\
&- \int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\chi_{\widetilde{G}^c}\right)(z)\beta_q(z)\chi_G(z)g(z)dA(z)\\
&- \int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\beta_q\chi_{\widetilde{G}\setminus G}\right)(z)\chi_{\widetilde{G}}(z)g(z)dA(z)\\
&+ \int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\beta_q(z)\chi_{\widetilde{G}\setminus G}(z)g(z)dA(z)\\
&+ \int_{\mathbb{D}}\overline{b'(z)}\left[\mathbb{P}_s,\beta_q\right](b'\chi_{\widetilde{G}})\chi_{\widetilde{G}}(z)g(z)dA(z)\\
&:= (3_A)+(3_B)+(3_C)+(3_D)+(3_E).
\end{aligned}$$

We handle each of these terms separately.

2.4.1. *The Term* (3_A)*:* This is the easiest of the terms in (3). Note that by Hölder's inequality we arrive at

$$\begin{aligned}
\left|(3_A)\right| &:= \left|\int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)g(z)\chi_{\widetilde{G}^c}(z)dA(z)\right|\\
&\leq \int_{\widetilde{G}^c}\left|b'(z)\right|\left|g(z)\right|\left|\mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\right|dA(z)\\
&\leq \left(\int_{\widetilde{G}^c}\left|b'(z)\right|^q\left|g(z)\right|^q dA_q(z)\right)^{1/q}\left(\int_{\mathbb{D}}\left|\mathbb{P}_s\left(b'\beta_q\chi_G\right)(z)\right|^p dA_p(z)\right)^{1/p}\\
&\lesssim \|b\|_{\mathcal{D}_q}^{1+\frac{q}{p}}\,\text{cap}_q\left(\cup_{j=1}^N I_j\right)\\
&\lesssim \|T_b\|^q\,\text{cap}_q\left(\cup_{j=1}^N I_j\right).
\end{aligned}$$

With the second to last line following from the fact that for $z\in\widetilde{G}^c$ we have $|g(z)|\lesssim\text{cap}_q\left(\cup_j I_j\right)$. We also used the fact that $\mathbb{P}_s$ is a bounded operator and similar computations to demonstrate that $f_q\in\mathcal{D}_p$.

2.4.2. *The Term* (3_B)*:* We need an estimate of

$$(3_B):=\int_{\mathbb{D}}\overline{b'(z)}\mathbb{P}_s\left(b'\chi_{\widetilde{G}^c}\right)(z)\beta_q(z)g(z)\chi_G(z)dA(z).$$

We first observe that, using the inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, we find

$$\begin{aligned}
|(3_B)| &\leq \int_{\mathbb{D}} \left| \overline{b'(z)} \mathbb{P}_s \left(b' \chi_{\widetilde{G}^c} \right)(z) \beta_q(z) g(z) \right| \chi_G(z) dA(z) \\
&\leq \epsilon \int_{\mathbb{D}} |b'(z)|^p \beta_q(z)^p \chi_G(z) dA_p(z) \\
&\quad + C(\epsilon) \int_{\mathbb{D}} \left| \mathbb{P}_s \left(b' \chi_{\widetilde{G}^c} \right)(z) \right|^q |g(z)|^q \chi_G(z) dA_q(z) \\
&\leq \epsilon \mu_{b,q} \left(\cup_j T(I_j) \right) + C(\epsilon) \int_{\mathbb{D}} \mathfrak{P}_s \left(|b'| \chi_{\widetilde{G}^c} \right)(z)^q |g(z)|^q \chi_G(z) dA_q(z).
\end{aligned}$$

The functions $\mathfrak{P}_s \left(|b'| \chi_{\widetilde{G}^c} \right)$ and $|g| \chi_G$ in the last integral have "disjoint" supports. Using this observation and a Schur-type argument, we claim the last integral is controlled by $C \|T_b\|^q \operatorname{cap}_q(\cup_j I_j)$. With this estimate, term (3_B) is then controlled by

$$|(3_B)| \leq \epsilon \mu_{b,q} \left(\cup_j T(I_j) \right) + C \|T_b\|^q \operatorname{cap}_q \left(\cup_j I_j \right),$$

which is what we needed to show.

2.4.3. *The Term* (3_C)*:* We next need to handle the following term:

$$(3_C) := \int_{\mathbb{D}} \overline{b'(z)} \mathbb{P}_s \left(b' \beta_q \chi_{\widetilde{G} \setminus G} \right)(z) \chi_{\widetilde{G}}(z) g(z) dA(z)$$

Using Hölder's Inequality we find that

$$\begin{aligned}
|(3_C)| &\leq \left(\int_{\mathbb{D}} |b'(z)|^q |g(z)|^q dA_q(z) \right)^{1/q} \left(\int_{\mathbb{D}} \left| \mathbb{P}_s \left(b' \beta_q \chi_{\widetilde{G} \setminus G} \right)(z) \right|^p dA_p(z) \right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{D}} |b'(z)|^q |g(z)|^q dA_q(z) \right)^{1/q} \left(\int_{\mathbb{D}} \left| b'(z) \beta_q(z) \chi_{\widetilde{G} \setminus G}(z) \right|^p dA_p(z) \right)^{1/p} \\
&= \left(\int_{\mathbb{D}} |b'(z)|^q |g(z)|^q dA_q(z) \right)^{1/q} \left(\mu_{b,q} \left(\widetilde{G} \setminus G \right) \right)^{1/p}.
\end{aligned}$$

We can arrange the enlargement $\widetilde{G}$ so that we additionally have the property

$$\mu_{b,q} \left(\widetilde{G} \setminus G \right) \leq \epsilon \mu_{b,q}(G).$$

Also, using the arguments related to term (4) we have that

$$\int_{\mathbb{D}} |b'(z)|^q |g(z)|^q dA_q(z) \leq \epsilon \left(\mu_{b,q} \left(\cup_j I_j \right) + C \|T_b\|^q \operatorname{cap}_q \left(\cup_j I_j \right) \right).$$

Using these estimates and the inequality that $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, we have

$$|(3_C)| \leq C \epsilon \mu_{b,q} \left(\cup_j I_j \right) + C \|T_b\|^q \operatorname{cap}_q \left(\cup_j I_j \right),$$

which is the estimate that we seek.

2.4.4. *The Term* (3_D)*:* We now handle the term

$$(3_D) := \int_{\mathbb{D}} \overline{b'(z)}\mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\beta_q(z)\chi_{\widetilde{G}\setminus G}(z)g(z)dA(z).$$

This is one of the easier that we have to estimate. Using Hölder's Inequality we see that

$$\begin{aligned}
|(3_D)| &\leq \left(\int_{\widetilde{G}\setminus G}\left|b'(z)\beta_q(z)\right|^p dA_p(z)\right)^{1/p}\left(\int_{\mathbb{D}}\left|\mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\right|^q\chi_{\widetilde{G}\setminus G}(z)dA_q(z)\right)^{1/q}\\
&\lesssim \left(\epsilon\mu_{b,q}\left(G\right)\right)^{1/p}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}\\
&\leq \left(\epsilon\mu_{b,q}\left(G\right)\right)^{1/p}\left((1+\epsilon)\mu_{b,q}\left(G\right)\right)^{1/q} = \epsilon^{1/p}(1+\epsilon)^{1/q}\mu_{b,q}\left(G\right).
\end{aligned}$$

But, this is an acceptable term since for ϵ chosen sufficiently small, we can hide this term back on the left hand side of the main estimate.

2.4.5. *The Term* (3_E)*:* Here we consider the term which vanishes when $q = 2$:

$$(3_E) \equiv \int_{\mathbb{D}} \overline{b'(z)}\left[\mathbb{P}_s, \beta_q\right]\left(b'\chi_{\widetilde{G}}\right)(z)g(z)\chi_{\widetilde{G}}(z)dA(z). \tag{2.4}$$

We wish to obtain an estimate for the commutator $[\mathbb{P}_s, \beta_q]$ that is better than the estimates for the operators $\mathbb{P}_s\beta_q$ and $\beta_q\mathbb{P}_s$ individually. Computing, we see

$$\begin{aligned}
\overline{b'(z)}\chi_{\widetilde{G}}(z)\left[\mathbb{P}_s, \beta_q\right]\left(b'\chi_{\widetilde{G}}\right)(z) &= \left(\mathbb{P}_s\left(\beta_q\left(b'\chi_{\widetilde{G}}\right)\right)(z) - \beta_q\left(z\right)\mathbb{P}_s\left(b'\chi_{\widetilde{G}}\right)(z)\right)\overline{b'(z)}\chi_{\widetilde{G}}(z)\\
&= c_s\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^s}{\left(1-z\overline{w}\right)^{s+2}}\left\{\beta_q\left(w\right)-\beta_q\left(z\right)\right\}b'(w)\overline{b'(z)}\chi_{\widetilde{G}}(w)dA(w)\chi_{\widetilde{G}}(z).\\
&= \left[\mathbb{P}_s, \beta_q\overline{b'}\,\right](b'\chi_{\widetilde{G}})(z)\chi_{\widetilde{G}}(z) + \left[\mathbb{P}_s, \overline{b'}\,\right](\beta_q b'\chi_{\widetilde{G}})(z)\chi_{\widetilde{G}}(z).
\end{aligned}$$

Key to the rest of the argument is the following Lemma. Define the following norm on functions (not necessarily analytic) by

$$\|\gamma\|_{\mathfrak{B}} := \sup_{z\in\mathbb{D}}(1-|z|^2)|\nabla\gamma(z)|.$$

Note that if γ is analytic then we have $\|\gamma\|_{\mathfrak{B}} = \|b\|_{\mathcal{B}}$, where $\mathcal{B}(\mathbb{D})$ is the Bloch space.

Lemma 1. *For* $1 < p < \infty$ *we have that*

$$[\mathbb{P}_s, \gamma] : L^p(\mathbb{D}, dA_p) \to L^p(\mathbb{D}, dA_p)$$

with $\|[\mathbb{P}_s, \gamma]\|_{L^p(dA_p)\to L^p(dA_p)} \lesssim \|\gamma\|_{\mathfrak{B}}$.

Assume Lemma 1 for the moment. With this we can conclude the estimate of (3_E). To do this, we proceed as follows.

$$\begin{aligned}
|(3_E)| &\leq \int_{\mathbb{D}} |g(z)| \left| \left[\mathbb{P}_s, \beta_q \overline{b'}\right] (b'\chi_{\widetilde{G}})(z)\right| \chi_{\widetilde{G}}(z) dA(z) + \int_{\mathbb{D}} |g(z)| \left| \left[\mathbb{P}_s, \overline{b'}\right] (\beta_q b'\chi_{\widetilde{G}})(z)\right| \chi_{\widetilde{G}}(z) dA(z) \\
&\leq \left(\int_{\mathbb{D}} |g(z)|^q \, dA_q(z)\right)^{1/q} \left(\int_{\mathbb{D}} \left| \left[\mathbb{P}_s, \beta_q \overline{b'}\right] (b'\chi_{\widetilde{G}})(z)\right|^p dA_p(z)\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{D}} |g(z)|^q \, dA_q(z)\right)^{1/q} \left(\int_{\mathbb{D}} \left| \left[\mathbb{P}_s, \overline{b'}\right] (\beta_q b'\chi_{\widetilde{G}})(z)\right|^p dA_p(z)\right)^{1/p} \\
&\leq C \|g\|_{\mathcal{D}_q} \left[\left\|\beta_q \overline{b'}\right\|_{\mathfrak{B}} \left(\int_{\mathbb{D}} |b'(z)|^p \chi_{\widetilde{G}}(z) dA_p(z)\right)^{1/p} + \left\|\overline{b'}\right\|_{\mathfrak{B}} \left(\int_{\mathbb{D}} |\beta_q(z) b'(z)|^p \chi_{\widetilde{G}}(z) dA_p(z)\right)^{1/p} \right] \\
&\leq C \operatorname{cap}_q(G) \left[\|b\|_{\mathcal{B}}^{q-1} \|b\|_{\mathcal{D}_p} + \|b\|_{\mathcal{B}} \|b\|_{\mathcal{D}_q}^{q-1} \right] \\
&\leq C \operatorname{cap}_q(G) \|T_b\|^q .
\end{aligned}$$

This is the estimate that we seek. In the course of the proof above, we used that dA_p is a $\mathcal{D}_p$-Carleson measure. This follows from the observation that for any compact subset E of the boundary $\mathbb{T}$ we have

$$\int_{T(E)} dA_p(z) \lesssim \operatorname{cap}_q(E),$$

which is the geometric characterization of the $\mathcal{D}_p$-Carleson measures.

We now wish to estimate the difference

$$\beta_q(w) - \beta_q(z) = \left|\left(1-|w|^2\right) b'(w)\right|^{q-2} - \left|\left(1-|z|^2\right) b'(z)\right|^{q-2},$$

and since $q > 2$, we first consider the difference

$$\left(1-|z|^2\right) b'(z) - \left(1-|w|^2\right) b'(w) = \mathfrak{D}b(z) - \mathfrak{D}b(w),$$

where $\mathfrak{D}b(z) = \left(1-|z|^2\right) b'(z)$ is the invariant derivative.

Let $\gamma(t)$ be the Bergman geodesic joining w to z, i.e. $\gamma : [0,1] \to \mathbb{D}$ with $\gamma(0) = w$ and $\gamma(1) = z$. Also, let $\beta(z,w)$ is the length between the points z and w measured in the Bergman or Poincaré metric in the unit disk. Then the fundamental theorem of calculus and the chain rule give

$$\begin{aligned}
\mathfrak{D}b(z) - \mathfrak{D}b(w) &= \int_0^1 \frac{d}{dt} \mathfrak{D}b(\gamma(t))\, dt \\
&= \int_0^1 \nabla(\mathfrak{D}b)(\gamma(t))\, \gamma'(t)\, dt.
\end{aligned}$$

For a function $h : \mathbb{D} \to \mathbb{C}$ define

$$Q_h(z) := \sup\left\{ \frac{|w\nabla h(z)|}{\langle B(z)w, w\rangle^{1/2}} : w \in \mathbb{C} \setminus \{0\} \right\}.$$

Here $B(z)$ is the matrix that gives rise to the Bergman metric at the point z. Continuing from above we have the following, upon taking absolute values we find,

$$\begin{aligned}|\mathfrak{D}b(z)-\mathfrak{D}b(w)| &= \left|\int_0^1 \nabla(\mathfrak{D}b)(\gamma(t))\gamma'(t)\,dt\right|\\ &\le \int_0^1 |\nabla(\mathfrak{D}b)(\gamma(t))\gamma'(t)|\,dt\\ &\le \int_0^1 Q_{\mathfrak{D}b}(\gamma(t))\langle B(\gamma(t))\gamma'(t),\gamma'(t)\rangle^{1/2}dt\\ &\le \beta(z,w)\sup_{\xi\in\mathbb{D}} Q_{\mathfrak{D}b}(\xi).\end{aligned}$$

The last line we have used the definition of the length $\beta(z,w)$. Next, one observes that $Q_h(z)=|\mathfrak{D}h(z)|$. This follows from the proof of the equivalences of (a), (b) and (e) in Theorem 3.1 of [7] (one can note that the analyticity plays no role in these equivalences). With this observation, we have the following,

$$|\mathfrak{D}b(z)-\mathfrak{D}b(w)| \le \beta(z,w)\sup_{\xi\in\mathbb{D}}|\mathfrak{D}(\mathfrak{D}b)(\xi)|.$$

Let $\mathcal{B}$ denote the Bloch space on the unit disc $\mathbb{D}$. This is the set of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f'(z)|.$$

Using the standard definition of the Bloch space and the equivalent norm,

$$\|f\|_{\mathcal{B}} \approx \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)^N\left|f^{(N)}(z)\right|$$

one sees that

$$\sup_{z\in\mathbb{D}}|\mathfrak{D}(\mathfrak{D}b)(z)| \lesssim \|b\|_{\mathcal{B}}.$$

Since $q>2$, and $\mathcal{D}_q\subset\mathcal{B}$ we then have the following estimate holding,

$$|\beta_q(z)-\beta_q(w)| \lesssim \beta(z,w)\|b\|_{\mathcal{B}}^{q-2} \lesssim \beta(z,w)\|b\|_{\mathcal{D}_q}^{q-2}.$$

We now substitute this estimate into the definition of the commutator and find the following:

$$\begin{aligned}\left|[\mathbb{P}_s,\beta_q]\left(b'\chi_{\widetilde{G}}\right)(z)\right| &\le c_s\|b\|_{\mathcal{D}_q}^{q-2}\int_{\mathbb{D}}\frac{\left(1-|w|^2\right)^s}{|1-z\overline{w}|^{s+2}}|b'(w)|\chi_{\widetilde{G}}(w)\beta(z,w)dA(w)\\ &=: c_s\|b\|_{\mathcal{D}_q}^{q-2}\mathfrak{P}_s\left(\beta(z,\cdot)|b'|\chi_{\widetilde{G}}\right)(z)\end{aligned}$$

Using Exercise 21 on page 79 of [7] we have the following Lemma at our disposal,

Lemma 2. *Let $-1<s$ and suppose $1<p<\infty$. Then the operator*

$$\mathfrak{S}_s f(z) := \int_{\mathbb{D}}\frac{\beta(z,w)\left(1-|w|^2\right)^s}{|1-z\overline{w}|^{2+s}}f(w)dA(w)$$

is bounded on $L^p(\mathbb{D};dA_p)$.

Then, one notes that $\mathfrak{P}_s\left(\beta(z,\cdot)\left|b'\right|\chi_{\widetilde{G}}\right)(z)=\mathfrak{S}_s\left(\left|b'\right|\chi_{\widetilde{G}}\right)(z)$ and we will use this Lemma to handle the second term.

We further substitute this estimate back into (3_E) given by (2.4). We thus see the following,

$$\begin{aligned}
|(3_E)| &\leq \int_{\mathbb{D}}\left|b'(z)\right|\left|[\mathbb{P}_s,\beta_q]\left(b'\chi_{\widetilde{G}}\right)(z)\right|\left|g(z)\right|\chi_{\widetilde{G}}(z)dA(z)\\
&\lesssim \|b\|_{\mathcal{D}_q}^{q-2}\int_{\mathbb{D}}\left|b'(z)\right|\left|g(z)\right|\mathfrak{S}_s\left(\left|b'\right|\chi_{\widetilde{G}}\right)(z)\chi_{\widetilde{G}}(z)dA(z)\\
&\leq \left(\int_{\mathbb{D}}\right)^{1/r}\left(\int_{\mathbb{D}}\right)^{1/s}\left(\int_{\mathbb{D}}\right)^{1/t}\\
&\leq \|b\|_{\mathcal{D}_q}^{q-2}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\left|g(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}\left(\int_{\mathbb{D}}\left(\mathfrak{S}_s\left(\left|b'\right|\chi_{\widetilde{G}}\right)(z)\right)^p\chi_{\widetilde{G}}(z)dA_p(z)\right)^{1/p}
\end{aligned}$$

Using estimate of the $\mathcal{D}_p$ and $\mathcal{D}_q$ norm of b by the norm of the bilinear form T_b, we arrive at

$$\begin{aligned}
|(3_E)| &\leq \|b\|_{\mathcal{D}_q}^{q-2}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\left|g(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}\left(\int_{\mathbb{D}}\left(\mathfrak{S}_s\left(\left|b'\right|\chi_{\widetilde{G}}\right)(z)\right)^p\chi_{\widetilde{G}}(z)dA_p(z)\right)^{1/p}\\
&\lesssim \|T_b\|^{q-2}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\left|g(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}\left(\int_{\mathbb{D}}\left|b'(z)\right|^p\chi_{\widetilde{G}}(z)dA_p(z)\right)^{1/p}\\
&\leq \|T_b\|^{q-2}\|b\|_{\mathcal{D}_p}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\left|g(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}\\
&\lesssim \|T_b\|^{q-1}\left(\int_{\mathbb{D}}\left|b'(z)\right|^q\left|g(z)\right|^q\chi_{\widetilde{G}}(z)dA_q(z)\right)^{1/q}.
\end{aligned}$$

We are left showing that this last integral can be estimated by the norm of the bilinear form T_b and the capacity of the collection of intervals.

2.5. **Term** (4)**:** This is one of the more challenging terms to handle. We first observe that by Hölder's inequality, we have that

$$\begin{aligned}
|(4)| &:= \left|\int_{\mathbb{D}}\overline{b'(z)}f_q(z)g'(z)dA(z)dA(z)\right|\\
&\leq 2\int_{\mathbb{D}}\left|b'(z)\right|\left|f_q(z)\right|\left|\varphi(z)\right|\left|\varphi'(z)\right|dA(z)\\
&\leq \epsilon\int_{\mathbb{D}}\left|b'(z)\right|^q\left|\varphi(z)\right|^q dA_q(z)+C(\epsilon)\int_{\mathbb{D}}\left|\varphi'(z)\right|^p\left|f_q(z)\right|^p dA_p(z)\\
&:= (4_A)+(4_B).
\end{aligned}$$

Now, consider term (4_A). We will use the properties of the extremal function φ to estimate this integral. We split the integral into three separate regions.

$$(4_A) = \epsilon\left\{\int_{\cup_j T(I_j)}+\int_{\cup_j T(\tilde{I}_j)\setminus\cup_j T(I_j)}+\int_{\left(\cup_j T(\tilde{I}_j)\right)^c}\right\}\left|b'(z)\right|^q\left|\varphi(z)\right|^q dA_q(z)$$

But, using the properties of the function φ, and that the collection of intervals $\{I_j\}$ was extremeal we arrive at

$$(4_A)\leq\epsilon\left(\mu_{b,q}\left(\cup_j I_j\right)+C\left\|T_b\right\|^q\operatorname{cap}_q\left(\cup_j I_j\right)\right)$$

As for term (4_B), we again note that the functions φ and f_q have disjoint supports. Applying the Schur argument, we can conclude that

$$(4_B) \leq \|T_b\|^q \operatorname{cap}_q (\cup_j I_j)$$

All together, we have

$$|(4)| \lesssim \epsilon \mu_{b,q} (\cup_j I_j) + \|T_b\|^q \operatorname{cap}_q (\cup_j I_j)$$

2.6. **The Term $|T_b(f_q, g)|$:**

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