ALCUNI ESERCIZI

(1) Sia $\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \geq 1;\ y^2+z^2 \leq 4,\ |x| \leq 3\}$.

(i) Trovare una parametrizzazione per $\partial\Omega$ e stabilire se essa sia compatibile con la normale ν uscente da Ω a $\partial\Omega$

(ii) Sia $F(x,y,z) = \left(\frac{x}{x^2+y^2+z^2},\ \frac{y}{x^2+y^2+z^2},\ 0\right) \in C^1(\mathbb{R}^3 \setminus \{(0,0,0)\})$

Calcolare il flusso di F attraverso (Σ, ν) dove $\Sigma = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ e $\nu(0,0,1) = (0,0,-1)$.

Svolgimento. $\partial\Omega = \overline{\Sigma}_a \cup \overline{\Sigma}_b \cup \overline{\Sigma}_{c+} \cup \overline{\Sigma}_{c-}$ con

$\overline{\Sigma}_a = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ ← le condizioni $|x| \leq 1$ e $y^2+z^2 \leq 4$ sono automatiche

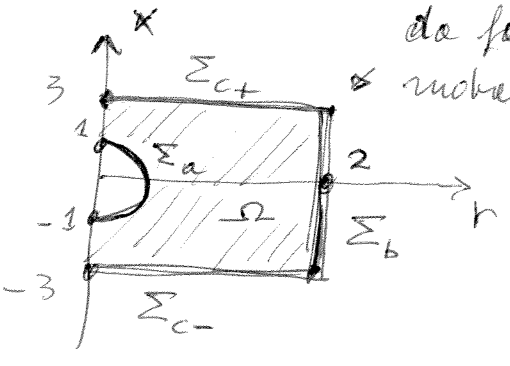
$\overline{\Sigma}_b = \{(x,y,z) : y^2+z^2 = 4;\ |x| \leq 3\}$ ← la condizione $x^2+y^2+z^2 \geq 1$ è automatica

$\overline{\Sigma}_{c+} = \{(x,y,z) : x = 3,\ y^2+z^2 \leq 4\}$ ← la condizione $x^2+y^2+z^2 \geq 1$ è automatica

$\overline{\Sigma}_{c-} = \{(x,y,z) : x = -3,\ y^2+z^2 \leq 4\}$ ← la condizione $x^2+y^2+z^2 \geq 1$ è automatica.

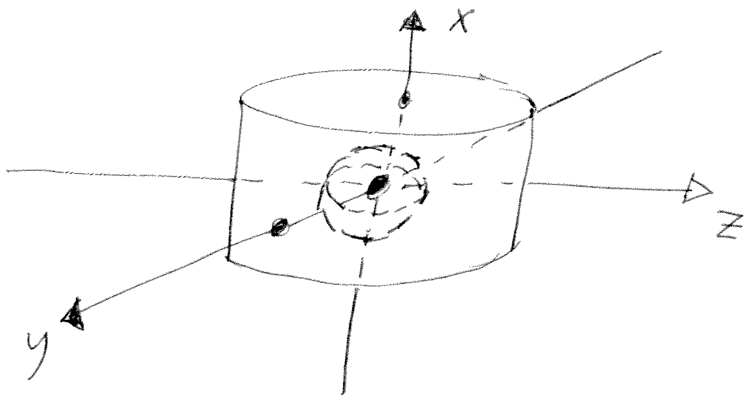
In coordinate cilindriche

$(*)\begin{cases} y = r\cos\theta \\ z = r\sin\theta \\ x = x \end{cases}$ $r \geq 0,\ |\theta| \leq \pi$, usando il fatto che $y^2+z^2 = r^2$,



da fare a mano.

Ω è un cilindro (un pezzo di cilindro) al cui interno è scavata una bolla:



Σ_a: Posso usare coordinate cartesiane, sferiche o cilindriche. Per coordinate cartesiane e cilindriche devo spezzare $\Sigma_a = \Sigma_{a+} \cup \Sigma_{a-}$

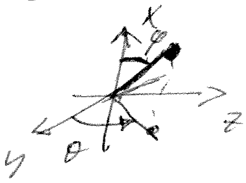
$\Sigma_{a+} = \{(x,y,z) \in \Sigma_a : x \geq 0\}$

$\Sigma_{a-} = \{(x,y,z) \in \Sigma_a : x \leq 0\}$

Uso quelle sferiche, per esempio:

$$\begin{cases} y = R \sin\varphi \cdot \cos\theta \\ z = R \sin\varphi \, \sin\theta \\ x = R \cos\varphi \end{cases} \qquad \text{con } R^2 = x^2+y^2+z^2 \overset{\text{su } \Sigma_a}{=} 1$$

$$0 \leq \varphi \leq \pi, \quad |\theta| \leq \pi.$$



$\Phi_a : [0,\pi] \times [-\pi,\pi] \longrightarrow \mathbb{R}^3$

$\Phi_a(\varphi,\theta) = (\cos\varphi, \sin\varphi \cdot \cos\theta, \sin\varphi \cdot \sin\theta) = (x,y,z)$

$\partial_\varphi \Phi_a = (-\sin\varphi, \cos\varphi \cdot \cos\theta, \cos\varphi \cdot \sin\theta)$

$\partial_\theta \Phi_a = (0, -\sin\varphi \cdot \sin\theta, \sin\varphi \cdot \cos\theta)$

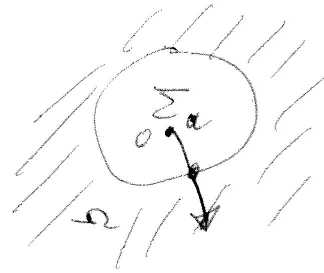
$$\partial_\varphi \Phi_a \times \partial_\theta \Phi_a = \begin{vmatrix} i & j & k \\ -\sin\varphi & \cos\varphi \cdot \cos\theta & \cos\varphi \cdot \sin\theta \\ 0 & -\sin\varphi \cdot \sin\theta & \sin\varphi \cdot \cos\theta \end{vmatrix}$$

$= i \cos\varphi \cdot \sin\varphi (\cos^2\theta + \sin^2\theta) + j \sin^2\varphi \cdot \cos\theta + k \sin^2\varphi \cdot \sin\theta$

$= i \cos\varphi \cdot \sin\varphi + j \sin^2\varphi \cdot \cos\theta + k \sin^2\varphi \cdot \sin\theta.$

Pongo $\varphi = \pi/2$, $\theta = 0$: $\Phi_a(\frac{\pi}{2}, 0) = (0,1,0)$

e $(\partial_\varphi \Phi_a \times \partial_\theta \Phi_a)(\frac{\pi}{2}, 0) = i\cdot 0 + j \cdot 1 + k \cdot 0 = (0,1,0)$



Il verso di $\partial_\varphi \Phi_a \times \partial_\theta \Phi_a$ punta verso l'<u>interno</u> di Ω:
Φ_a non è compatibile.

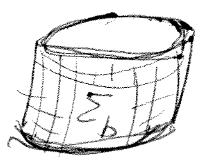
$$\Phi_a : [0,\pi] \times [-\pi, \pi] \longrightarrow \mathbb{R}^3$$

$$\Phi_a(\varphi, \theta) = (\cos\varphi, \sin\varphi \cdot \cos\theta, \sin\varphi \cdot \sin\theta)$$

$\partial_\varphi \Phi_a \times \partial_\theta \Phi_a = (\cos\varphi \cdot \sin\varphi, \sin^2\varphi \cdot \cos\theta, \sin^2\varphi \cdot \sin\theta)$ non è compatibile con ν.

Σ_b Uso le coordinate cilindriche (*):

$$r^2 = y^2 + z^2 = 4$$
$$|x| \le 3$$



$$\begin{cases} y = 2\cdot\cos\theta \\ z = 2\cdot\sin\theta \\ x = x \end{cases}$$

$$\Phi_b : [-\pi, \pi] \times [-3, 3] \longrightarrow \mathbb{R}^3$$

$$\Phi_b(\theta, x) = (x, 2\cos\theta, 2\sin\theta)$$

$\partial_\theta \Phi_b \times \partial_x \Phi_b =$

~~$= (-2\sin\theta, 2\cos\theta, 0) \times (0,0,1) =$~~ $\begin{vmatrix} i & j & k \\ -2\sin(\theta) & 2\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} =$

$= (0, -2\sin\theta, 2\cos\theta) \times (1,0,0) = \begin{vmatrix} i & j & k \\ 0 & -2\sin\theta & 2\cos\theta \\ 1 & 0 & 0 \end{vmatrix} =$

$= (0, +2\cos\theta, 2\sin\theta)$

$\partial_\theta \Phi_b \times \partial_x \Phi_b = (0, 2\cos\theta, 2\sin\theta)$
è compatibile con ν

Per $\theta = 0$:
$\Phi_b(0,x) = (x, 2, 0)$
$\partial_\theta \Phi_b \times \partial_x \Phi_b = (0,2,0)$

Uso le parametrizzazioni

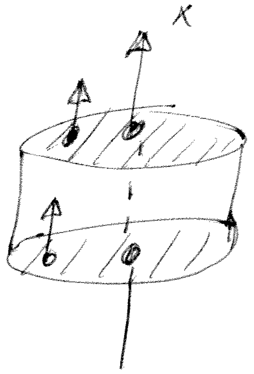
$\Phi_{c+}, \Phi_{c-} : \{(y,z) : y^2+z^2 \le 4\} = B(0,2) \longrightarrow \mathbb{R}^3$

$\Phi_{c+}(y,z) = (3,y,z)$, $\Phi_{c-}(y,z) = (-3,y,z)$

$\partial_y \Phi_{c\pm} \times \partial_z \Phi_{c\pm} = (0,1,0)\times(0,0,1) = \begin{vmatrix} i & j & k\\ 0&1&0\\0&0&1\end{vmatrix}$

$= (1,0,0)$

Φ_{c+} è compatibile con ν, Φ_{c-} non lo è.



Calcolo ora il flusso. Osservo che

$(\Sigma,\nu) = (\Sigma_a, \text{normale esterna})$.

Non m'è necessario usare il Teorema della divergenza. Usando la parametrizzazione Φ_a:

$$\iint_\Sigma F\cdot\nu\, d\sigma = -\int_0^\pi d\varphi \cdot \int_{-\pi}^{\pi} d\theta \cdot \left[F(\Phi_a(\varphi,\theta))\cdot \partial_\varphi\Phi_a(\varphi,\theta)\times\partial_\theta\Phi_a(\varphi,\theta)\right]$$

$$= -\int_0^\pi d\varphi \cdot \int_{-\pi}^\pi d\theta \cdot (\cos\varphi, \sin\varphi\cdot\cos\theta, 0)\cdot(\cos\varphi\cdot\sin\varphi, \sin^2\varphi\cdot\cos\theta, \sin^2\varphi\cdot\sin\theta)$$

$$= -\int_0^\pi \cos^2\varphi\cdot\sin\varphi\,d\varphi \cdot \int_{-\pi}^\pi 1\,d\theta - \int_{-\pi}^\pi\cos^2\theta\,d\theta\cdot\int_0^\pi \sin^3\varphi\,d\varphi,$$

che calcolo usando: $\int_{-\pi}^\pi d\theta = 2\pi$; $\int_{-\pi}^\pi\cos^2\theta\,d\theta=\pi$;

$$\int_0^\pi \cos^2\varphi\cdot\sin\varphi\,d\varphi = -\int_0^\pi\cos^2\varphi\cdot d\cos\varphi = -\int_1^{-1} v^2\, dv = \frac{v^3}{3}\Big|_{-1}^{1} = \frac23$$

$$\int_0^\pi \sin^3\varphi\, d\varphi = \int_0^\pi (1-\cos^2\varphi)\sin\varphi\, d\varphi = \int_0^\pi\sin\varphi\, d\varphi - \int_0^\pi\cos^2\varphi\cdot\sin\varphi\, d\varphi$$

$$= 2-\frac23 = \frac43$$

$\boxed{\text{Flusso} = -8/3\cdot\pi}$

(2) Risolvere
$$\begin{cases} \ddot{x} - \dot{x} + x = e^{t/2} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) & (E) \\ x(0) = 1 \\ \dot{x}(0) = 0 \end{cases} \quad \Big\} (CI)$$

Svolgimento. Considero l'omogenea (E0) associata a (E):
$$(E0) \quad \ddot{z} - \dot{z} + z = 0$$
avente equazione caratteristica $\lambda^2 - \lambda + 1 = 0$

con soluzioni $\lambda = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$.

L'integrale generale di (E0) è
$$z(t) = e^{\frac{t}{2}} \cdot \left[A \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) + B \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right], \quad A, B \in \mathbb{R}.$$

Cerco ora una soluzione particolare di (E).

Poiché il termine noto $f(t) = e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right)$

ho "risonanza"; cerco $\bar{x}(t)$ nella forma
$$\bar{x}(t) = t \cdot e^{\frac{t}{2}} \left[A \cos\left(\frac{\sqrt{3}}{2} t\right) + B \sin\left(\frac{\sqrt{3}}{2} t\right)\right]$$
$$\dot{\bar{x}}(t) = e^{\frac{t}{2}} \left[A \cos\left(\frac{\sqrt{3}}{2} t\right) + B \sin\left(\frac{\sqrt{3}}{2} t\right)\right] +$$
$$+ t \cdot \left\{ \frac{1}{2} e^{\frac{t}{2}} \left[A \cos\left(\frac{\sqrt{3}}{2} t\right) + B \sin\left(\frac{\sqrt{3}}{2} t\right)\right] + e^{\frac{t}{2}} \cdot \left[-A \frac{\sqrt{3}}{2} \sin\left(\frac{\sqrt{3}}{2} t\right) + B \frac{\sqrt{3}}{2} \cos\left(\frac{\sqrt{3}}{2} t\right)\right] \right\} =$$
$$= e^{\frac{t}{2}} \cdot \left[A \cos\left(\frac{\sqrt{3}}{2} t\right) + B \sin\left(\frac{\sqrt{3}}{2} t\right)\right] +$$
$$+ t \cdot e^{\frac{t}{2}} \cdot \left[\left(\frac{1}{2} A + \frac{\sqrt{3}}{2} B\right) \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) + \left(\frac{1}{2} B - \frac{\sqrt{3}}{2} A\right) \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right]$$

Allo stesso modo calcolo:

$$\ddot{x}(t) = \cancel{e^{\frac{t}{2}} \cdot A\cos\left(\frac{\sqrt{3}}{2}t\right.}$$

$$= e^{\frac{t}{2}} \cdot 2 \cdot \left[\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right)\cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{1}{2}B - \frac{\sqrt{3}}{2}A\right)\cdot \sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$+ t \cdot \left\{ \frac{1}{2} e^{\frac{t}{2}} \cdot \left[\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right)\cdot \cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{1}{2}B - \frac{\sqrt{3}}{2}A\right)\cdot \sin\left(\frac{\sqrt{3}}{2}t\right)\right]\right.$$

$$\left. + e^{\frac{t}{2}} \cdot \left[-\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right)\cdot \sin\left(\frac{\sqrt{3}}{2}t\right)\cdot\frac{\sqrt{3}}{2} + \left(\frac{1}{2}B - \frac{\sqrt{3}}{2}A\right)\frac{\sqrt{3}}{2}\cdot \cos\left(\frac{\sqrt{3}}{2}t\right)\right]\right\},$$

Sostituendo in (E) e sapendo ~~da~~ a priori che i termini con il fattore t scompaiono (verificare!) ho:

$$\ddot{x} - \dot{x} + x = e^{\frac{t}{2}} \cdot \left\{ \underbrace{2\left[\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right)\cos\left(\frac{\sqrt{3}}{2}t\right) + \left(\frac{1}{2}B - \frac{\sqrt{3}}{2}A\right)\cdot \sin\left(\frac{\sqrt{3}}{2}t\right)\right]}_{\ddot{x}} \right.$$

$$\left. \underbrace{- \left[A\cdot\cos\left(\frac{\sqrt{3}}{2}t\right) + B\sin\left(\frac{\sqrt{3}}{2}t\right)\right]}_{-\dot{x}} + \underset{x}{\underset{\shortparallel}{0}} \right\}$$

$$= e^{t/2} \cdot \left[2\cdot\left(\frac{1}{2}A + \frac{\sqrt{3}}{2}B\right) - A\right]\cdot\cos\left(\frac{\sqrt{3}}{2}t\right) + e^{\frac{t}{2}} \cdot \left[2\cdot\left(\frac{1}{2}B - \frac{\sqrt{3}}{2}A\right) - B\right]\cdot \sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$= e^{\frac{t}{2}} \cdot \left[\sqrt{3}B\cdot\cos\left(\frac{\sqrt{3}}{2}t\right) - \sqrt{3}A\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$$\overset{*}{=} f(t) = e^{t/2} \cdot \cos\left(\frac{\sqrt{3}}{2}t\right) \iff A = 0,\quad B = \frac{1}{\sqrt{3}}.$$

Quindi

$\bar{x}(t) = \dfrac{t\,e^{t/2}}{\sqrt{3}} \cdot \cos\left(\dfrac{\sqrt{3}}{2}t\right)$ è una soluzione di (E), mentre l'integrale generale di (E) è:

$$x(t) = \frac{t\cdot e^{t/2}}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) + e^{\frac{t}{2}}\left[A\cdot\cos\left(\frac{\sqrt{3}}{2}t\right) + B\cdot\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

$A, B \in \mathbb{R}$.

Quindi $1 = x(0) = A + B$

Calcolo $\dot{x}(t) = \frac{e^{t/3}}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) + \frac{t}{\sqrt{3}} \cdot \left[\frac{1}{2} e^{\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) + e^{\frac{t}{2}} \cdot \frac{\sqrt{3}}{2} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right)\right] + \frac{1}{2} \cdot e^{\frac{t}{2}} \left[A \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) + B \cdot \sin\left(\frac{\sqrt{3}}{2} t\right)\right]$

$+ e^{\frac{t}{2}} \cdot \frac{\sqrt{3}}{2} \cdot \left[-A \cdot \sin\left(\frac{\sqrt{3}}{2} t\right) + B \cos\left(\frac{\sqrt{3}}{2} t\right)\right]$,

quindi

$$0 = \dot{x}(0) = \frac{1}{2} A + \frac{\sqrt{3}}{2} B$$

$$\begin{cases} A + B = 1 \\ A + \sqrt{3} B = 0 \end{cases} \quad \begin{cases} A = -\sqrt{3} B \\ (1 - \sqrt{3}) B = 1 \end{cases} \quad \begin{cases} B = -1/(\sqrt{3} - 1) \\ A = \sqrt{3}/(\sqrt{3} - 1) \end{cases}$$

$$\boxed{x(t) = \frac{t}{\sqrt{3}} e^{t/2} \cdot \sin\left(\frac{\sqrt{3}}{2} t\right) + \frac{e^{t/2}}{\sqrt{3} - 1} \cdot \left[\sqrt{3} \cdot \cos\left(\frac{\sqrt{3}}{2} t\right) - \sin\left(\frac{\sqrt{3}}{2} t\right)\right]}$$

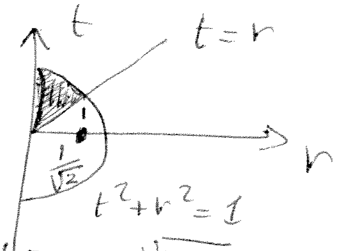
3) $\Omega = \{(x, y, z) : z \geq \sqrt{x^2 + y^2} ; \; x^2 + y^2 + z^2 \leq 1\}$.

$$\text{Volume}(\Omega) = \iiint_{\Omega} dx\, dy\, dz \doteq$$

Uso coordinate cilindriche $\begin{cases} x = r \cos\theta \\ y = r \sin\theta \\ z = t \end{cases}$ $dx\, dy\, dz = r \cdot dr\, d\theta\, dt$

$|\theta| \leq \pi$; $r \geq 0$; $t \in \mathbb{R}$

$t \geq r, \quad t^2 + r^2 \leq 1$



$\begin{cases} t = r \\ t^2 + r^2 = 1 \end{cases}$ $\quad 2r^2 = 1, \quad r = 1/\sqrt{2}$

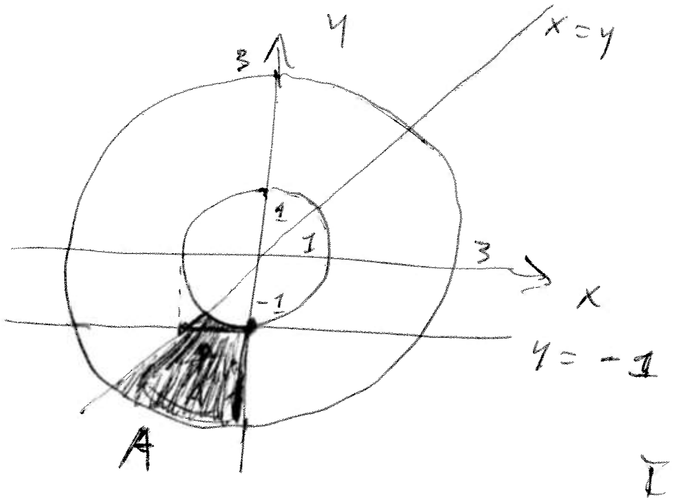
$$\doteq \int_{-\pi}^{\pi} d\theta \int_0^{1/\sqrt{2}} r\, dr \int_r^{\sqrt{1 - t^2}} dt = 2\pi \cdot \int_0^{1/\sqrt{2}} \left(\sqrt{1 - r^2} - r\right) \cdot r\, dr$$

$$= 2\pi \cdot \left\{\frac{1}{2} \int_0^{1/\sqrt{2}} \sqrt{1 - r^2}\, dr^2 - \int_0^{1/\sqrt{2}} r^2 dr\right\} = 2\pi \left\{\frac{1}{2} \left[\frac{(1 - r^2)^{3/2}}{3/2}\right]_0^{1/\sqrt{2}} - \left[\frac{r^3}{3}\right]_0^{1/\sqrt{2}}\right\}$$

$$= 2\pi \cdot \left\{ \frac{-1}{3}\left[\left(\frac{1}{2}\right)^{3/2} - 1\right] - \frac{1}{3}\left(\frac{1}{2^{1/2}}\right)^3 \right\} = \frac{2}{3}\pi\left(1 - \frac{2}{\sqrt{2}^3}\right)$$

$$= \frac{2}{3}\pi \cdot \left(1 - \frac{1}{\sqrt{2}}\right).$$

(4) $I = \iint_A \frac{2x^2 - 3y^2}{x^2+y^2}\, dx\, dy \quad A = \{1 \le x^2+y^2 \le 9,\ y \le x \le 0\}$



se metto $y \ge -1$, ho un integrale fattibile, ma complicato.

In coordinate polari

$$I = \int_1^3 r\,dr \int_{5/4\pi}^{6/4\pi} d\theta \cdot (2\cos^2\theta - 3\sin^2\theta)\,d\theta$$

$$= 4 \cdot \int_{5/4\pi}^{3/2\pi} \left(2 \cdot \frac{\cos(2\theta)-1}{2} - 3\,\frac{1-\cos(2\theta)}{2}\right) d\theta$$

$$= \int_{5/2\pi}^{3\pi} \left[2(\cos(v)-1) - 3(1-\cos(v))\right] dv$$

$$= \int_{5/2\pi}^{3\pi} (5\cos(v) - 5)\,dv = -5\cdot\left(3 - \frac{5}{2}\right)\pi + 5\sin(v)\Big|_{5/2\pi}^{3\pi}$$

$$= -\frac{5}{2}\pi + 5(0-1) = -5\left(\frac{\pi}{2}+1\right)$$

$$\boxed{I = -5\cdot\left(\frac{\pi}{2}+1\right)}$$