From Grushin to Heisenberg via an isoperimetric problem *

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Abstract

The Grushin plane is a right quotient of the Heisenberg group. Heisenberg geodesics' projections are solutions of an isoperimetric problem in the Grushin plane.

1 Introduction

It is a known fact that there is a correspondence between isoperimetric problems in Riemannian surfaces and sub-Riemannian geometries in three-dimensional manifolds. The most significant example is the isoperimetric problem in the plane, corresponding to the sub-Riemannian geometry of the Heisenberg group $\mathbb{H}$.

We briefly recall this connection following the exposition in [Mont]. Consider, on the Euclidean plane, the one-form $\alpha = \frac{1}{2}(xdy - ydx)$, which satisfies $d\alpha = dx \wedge dy$ and which vanishes on straight lines through the origin. By Stokes' Theorem, the signed area enclosed by a curve γ is $\int_\gamma \alpha$. Let $c : [a, b] \to \mathbb{R}^2$ be a curve. For each s in $[a, b]$, let γ_s be the union of the curve c restricted to $[a, s]$, of the segment of straight line joining $c(s)$ with the origin O and of the segment of straight line joining O with $c(a)$. Let $C : [a, b] \to \mathbb{R}^3$ be the curve

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$C(s) = (c(s), \int_{\gamma_s} \alpha)$. The third coordinate of $C(s)$ is the signed area enclosed by γ_s. The curve $C = (x, y, t)$ satisfies the O.D.E.

$$dt = \frac{1}{2}(xdy - ydx).$$

A path C in three-dimensional space which is obtained in this way is called a *horizontal lift* of the curve c. More generally, a *horizontal curve* is any curve $C = (x, y, t)$ which satisfies the O.D.E. above. A notion of length for horizontal curves $C = (x, y, t)$ is defined by setting

$$\Lambda_{\mathbb{H}}(C) = \int_C \sqrt{dx^2 + dy^2},$$

i.e. the length $\Lambda_{\mathbb{H}}(C)$ of C is the Euclidean length of C's projection onto the (x, y)-plane. Given points P and Q in $\mathbb{R}^3$, we define their *Carnot-Charathéodory distance* $d_{\mathbb{H}}(P, Q)$ as the infimum of $\Lambda_{\mathbb{H}}(C)$ as C ranges over the set $\mathcal{C}(P, Q)$ of the horizontal curves joining P and Q. It is easy to see that $\mathcal{C}(P, Q)$ is not empty (alternatively, this is a special case of the non-elementary Chow's Theorem [Mont]). The *Heisenberg group*, as metric space, is $(\mathbb{R}^3, d_{\mathbb{H}})$. In this context, we write $\mathbb{H} = \mathbb{R}^3$.

More geometrically, given a horizontal curve C between two points in $\mathbb{H}$, $P = (x_1, y_1, t_1)$ and $Q = (x_2, y_2, t_2)$, and considering its projection c on the (x, y)-plane, we have that (i) $\Lambda_{\mathbb{H}}(C) = \Lambda_{\mathrm{Euc}}(c)$ is the Euclidean length of c and (ii) $A = t_2 - t_1$ is the signed area enclosed between c and the straight segment between (x_2, y_2) and (x_1, y_1). A length minimizing horizontal curve Γ between P and Q, a *geodesic* of $(\mathbb{H}, d_{\mathbb{H}})$, is then the horizontal lift of a plane curve γ between (x_1, y_1) and (x_2, y_2) such that (i) γ, together with the segment between (x_2, y_2) and (x_1, y_1), encloses an area A and (ii) γ has minimal Euclidean length among the curves with property (i). This is (a signed version of) the classical Dido problem in the plane, and it is well known that its unique solution γ is an arc of a circle. Hence, the geodesics in $(\mathbb{H}, d_{\mathbb{H}})$ are horizontal lifts of circular arcs.

As an algebraic object, the *Heisenberg group* $\mathbb{H}$ is the Lie group $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ endowed with the product

$$(z, t) \cdot (w, s) = (z + w, t + s - 1/2 Im(z\overline{w})).$$

The metric $d_{\mathbb{H}}$ on $\mathbb{H}$ is left invariant: $d_{\mathbb{H}}(A \cdot P, A \cdot Q) = d_{\mathbb{H}}(P, Q)$, whenever P, Q, $A \in \mathbb{H}$. When no confusion arises, we drop the dot "$\cdot$" in the product. In Section 2 we give a less euristic definition of $d_{\mathbb{H}}$.

The *Grushin plane* $\mathbb{G}$ is the plane $\mathbb{R}^2$ with coordinates (u, v), endowed, outside the *critical line* $u = 0$, with the Riemannian metric

$$ds^2 = du^2 + \frac{dv^2}{u^2}.$$

The metric can be extended continuously across the critical line to a metric $d_{\mathbb{G}}$ on $\mathbb{R}^2$. We write $\mathbb{G} = \mathbb{R}^2$, when the latter is endowed with the metric $d_{\mathbb{G}}$. We refer the reader to [FL] for a presentation of $\mathbb{G}$. In Section 2 we will recall some elementary properties of Grushin geometry.

It is known that the Grushin plane $\mathbb{G}$, as metric space, is a quotient of the Heisenberg group, hence, in principle, there is a relationship between the sub-Riemannian geometry in $\mathbb{H}$ and a specific isoperimetric problem in $\mathbb{G}$. In this note, we investigate more closely this relationship. Indeed, we consider the following two Dido-type problems (see Theorem 3).

Isoperimetric Problem A *Consider* $\xi = (a, 0)$ $\eta = (b, 0)$ *in* $\mathbb{G}$, $0 < a \leq b$. *Given* $0 < A < +\infty$, *find a bounded open set* Ω *in* $\mathbb{G}$ *having as boundary an absolutely continuous curve* γ *from* ξ *to* η *and the straight line from* η *to* ξ, *such that*

(1)

$$\int_\Omega \frac{dudv}{u^2} = A,$$

(2) γ *has minimal length with respect to the Grushin geometry.*

Isoperimetric Problem B *Consider* $\xi = (a, 0)$ $\eta = (b, 0)$ *in* $\mathbb{G}$, $0 < a \leq b$. *Given* $0 < A < +\infty$, *find an absolutely continuous curve* γ *from* ξ *to* η *and the straight line from* η *to* ξ, *for which*

(1)

$$-\int_\gamma \frac{dv}{u} = A,$$

(2) γ *has minimal length with respect to the Grushin geometry.*

Problem A is a *bona fide* isoperimetric problem, while Problem B is an "isoperimetric problem for signed areas", which is directly related to the geodesics in $\mathbb{H}$. On a formal level the two problems are equivalent by Gauss-Green formula and we are interested in verifying to what extent they admit the same solution.

The solution of problems A and B emphasizes the relationship between $\mathbb{H}$ and $\mathbb{G}$. We can view $\mathbb{G}$ as a quotient of $\mathbb{H}$, in the following sense. Consider the one-parameter subgroup $\mathcal{S} = \{(\tau, 0, 0) : \tau \in \mathbb{R}\} \subset \mathbb{H}$. Consider $\mathbb{X} = \mathcal{S}\diagdown\mathbb{H} = \{\mathcal{S}P : P \in \mathbb{H}\}$, the class of the right cosets of $\mathcal{S}$, endowed with the quotient metric

$$d_{\mathbb{X}}(\mathcal{S}P, \mathcal{S}Q) = \inf_{H,K\in\mathcal{S}} d_{\mathbb{H}}(HP, KQ). \tag{1}$$

By left invariance, $d_{\mathbb{X}}(\mathcal{S}P, \mathcal{S}Q) = \inf_{H\in\mathcal{S}} d_{\mathbb{H}}(HP, Q)$.

In Theorem 1 below we show that $\mathbb{X}$ is isometric to $\mathbb{G}$. The isometry can be realized as follows. We consider the function $\Phi : \mathbb{H} \to \mathbb{G}$,

$$(u, v) = \Phi(x, y, t) := (y, t - \frac{xy}{2}).$$

Let $P = (x, y, t)$, the function Φ identifies each coset $\mathcal{S}P = \{(\tau, 0, 0)(x, y, t) : \tau \in \mathbb{R}\} = \{(\tau + x, y, t + 1/2\tau y) : \tau \in \mathbb{R}\}$ with $(0, u, v) = \mathcal{S}P \cap \{x = 0\}$, its intersection with the plane $x = 0$ in $\mathbb{H}$.

The relationship between $\mathbb{H}$ and $\mathbb{G}$ was already pointed out by Rotschild and Stein ([RS]), who observed that, adding a dummy variable, two vector fields spanning the horizontal space of a Grushin plane could be seen as the vector fields spanning the horizontal space of the Heisenberg group. In their paper [RS], they developed this observation as far as proving that any family of vector fields satisfying Hörmander's condition, by adding new variables and using an approximation similar to the Euclidean approximation of differentiable manifolds, locally leads to a nilpotent Lie group. This enabled them to use geometric and analytic tools from Lie group theory in the study of general Hörmander's vector fields.

The isoperimetric problems A and B are different from the isoperimetric problem in the Grushin plane recently solved by Monti and Morbidelli in [MM]. Generalizing the results in [Beck], they solve the isoperimetric problem for a class of Grushin-like structures. Let $\alpha \geq 0$. For smooth domains Ω in the (u, v)-plane, the problem they consider is that of minimizing the functional

$$\int_{\partial\Omega} (\dot{v}^2(t) + u^{2\alpha}(t)\dot{u}^2(t))^{1/2} dt$$

over the domains such that $\int_{\Omega} dudv = A$ is fixed. The isoperimetric problem we have considered does not belong to this family.

The relationship between $\mathbb{H}$ and $\mathbb{G}$ is explained in Theorem 1 below. The isoperimetric Problem B is solved in Corollary 2, and the isoperimetric Problem A is solved in Theorem 3.

2 Heisenberg and Grushin

Among the left-invariant metrics on $\mathbb{H}$, we consider the *Carnot-Charathéodory metric* on $\mathbb{H}$. Let $z = x+iy$. Consider the left invariant vector fields

$$X = \frac{\partial}{\partial x} - 1/2y\frac{\partial}{\partial t}; \;\; Y = \frac{\partial}{\partial y} + 1/2x\frac{\partial}{\partial t}.$$

The vector fields X and Y do not commute. Indeed, $[X, Y] = \frac{\partial}{\partial t}$. This fact is central to the theory of Carnot groups, of which the Heisenberg group is the simplest nontrivial example.

An absolutely continuous curve (in the Euclidean sense) $\gamma : I \to \mathbb{H}$ is *horizontal* if $\dot{\gamma}(s) = a(s)X_{\gamma(s)} + b(s)Y_{\gamma(s)} \in Span\{X_{\gamma(s)}, Y_{\gamma(s)}\}$ for almost all $s \in I$. The space $H_P\mathbb{H} = Span\{X_P, Y_P\}$ is called the *horizontal space* at P. The $\mathbb{H}$-*length* of γ, $\Lambda_{\mathbb{H}}(\gamma)$, is the Euclidean length of γ's vertical projection onto the z-plane,

$$\Lambda_{\mathbb{H}}(\gamma) = \int_I \sqrt{a^2(\xi) + b^2(\xi)}d\xi.$$

Let P and Q be two points in $\mathbb{H}$. The *Carnot-Charathéodory distance* between P and Q, $d_{\mathbb{H}}(P, Q)$, is the infimum of the $\mathbb{H}$-lengths of the horizontal curves joining P and Q. Since the notion of horizontal curve and of $\mathbb{H}-$length are left invariant, the Carnot-Charathéodory distance is left invariant. Equivalently, $d_{\mathbb{H}}$ is the distance associated with the Carnot-Charathéodory metric making $\{X_P, Y_P\}$ into an orthonormal basis for $H_P\mathbb{H}$, for all $P \in \mathbb{H}$.

The Grushin plane $\mathbb{G}$ is endowed with the vector fields

$$U = \partial_u \quad \text{and} \quad V = -u\partial_v.$$

The *Grushin metric* outside the *critical line* $u = 0$ is the Riemannian metric ds^2 making U and V into a orthonormal basis for the tangent space,

$$ds^2 = du^2 + \frac{dv^2}{u^2}.$$

The metric can be extended across the critical line $u = 0$ as a Carnot-Charathèodory metric, since $[U, V] = -\partial_v \neq 0$. As already mentioned in the Introduction, by means of the length element ds^2 one can compute the $\mathbb{G}$-*length* $\Lambda_{\mathbb{G}}(\gamma)$ of a *horizontal curve* $\gamma : J \to \mathbb{G}$,

$$\Lambda_{\mathbb{G}}(\gamma) = \int_J \sqrt{\dot{u}^2(\xi) + \frac{\dot{v}^2(\xi)}{u^2(\xi)}}d\xi.$$

A curve is horizontal if it is absolutely continuous in the Euclidean sense and it has locally finite length with respect to the metric ds^2. In the usual way, the notion of $\mathbb{G}-$length leads to the geodesic metric $d_{\mathbb{G}}$.

The following Theorem explains the metric and algebraic relationship between $\mathbb{H}$ and $\mathbb{G}$.

Theorem 1 *Let $\mathcal{S}$ be a one-parameter subgroup of $\mathbb{H}$. If $\mathcal{S} = Z$ is the center of $\mathbb{H}$, then $\mathcal{S}\diagdown\mathbb{H}$ is isometric to the Euclidean plane $\mathbb{R}^2$. If $\mathcal{S}$ is any other one-parameter subgroup, then $\mathcal{S}\diagdown\mathbb{H}$ is isometric to $\mathbb{G}$.*

The center Z of $\mathbb{H}$ is the real line trough the origin generated by the vector field $\frac{\partial}{\partial t}$.

Proof. The case $\mathcal{S} = Z$ is trivial, the isometry being $f : (z,t) \mapsto z$. The case $\mathcal{S} \neq Z$ can be reduced to $\mathcal{S} = \{\mathcal{S}(\tau) = (\tau, 0, 0) : \tau \in \mathbb{R}\}$. Since the latter case is the one we are interested in, we omit the details of the reduction. Let $\mathcal{S} \cdot P \in \mathcal{S}\diagdown\mathbb{H}$ be a right translate of $\mathcal{S}$. The subgroup $\mathcal{S}$ has a unique intersection with the plane $\{x = 0\}$ in $\mathbb{H}$. Then, we can identify $\mathcal{S}\diagdown\mathbb{H}$ with $\mathbb{R}^2$, the identification being $\varphi : \mathcal{S} \cdot (0, u, v) \mapsto (u, v)$.

The map $\Psi : \mathbb{R}^3 \to \mathbb{H}$,

$$\Psi : (\tau, u, v) \mapsto \mathcal{S}(\tau)(0, u, v) = (\tau, u, v + 1/2\tau u) = (x, y, t), \qquad (2)$$

is an analytic change of variables in $\mathbb{H}$. We introduce new coordinates $[\cdot]$ in $\mathbb{H}$: $(x, y, t) = \Psi(\tau, u, v) =: [\tau, u, v]$. The natural projection $\pi : \mathbb{H} \to \mathcal{S}\diagdown\mathbb{H}$ becomes $\varphi \circ \pi : [\tau, u, v] \mapsto (u, v)$. Recall from the Introduction that $\mathbb{X}$ is $\mathcal{S}\diagdown\mathbb{H}$, endowed with the quotient distance (1).

The theorem is proved if the following holds.

Claim *The map $\varphi : \mathbb{X} \to \mathbb{G}$ is a surjective isometry.*

In the new variables, the vector fields X and Y become

$$X = \partial_\tau - u\partial_v \ , \quad Y = \partial_u \ . \qquad (3)$$

Their push-forward by π are

$$\varphi_*\pi_*X = -u\partial_v = V \ , \quad \varphi_*\pi_*Y = \partial_u = U \ .$$

By (1), to prove that $d_{\mathbb{X}}(\mathcal{S} \cdot P, \mathcal{S} \cdot Q) = d_{\mathbb{G}}(\varphi(\mathcal{S} \cdot P), \varphi(\mathcal{S} \cdot Q))$, for any $\mathcal{S} \cdot P, \mathcal{S} \cdot Q \in \mathcal{S}\diagdown\mathbb{H}$, it is enough to show that any horizontal curve in $\mathbb{G}$ has a *horizontal lift* in $\mathbb{H}$. Indeed, let $\gamma : [0, 1] \to \mathbb{G}$ be an horizontal curve. The curve $\tilde{\gamma} : [0, 1] \to \mathbb{H}$, is a horizontal lift of γ if it is a horizontal curve and $\varphi \circ \pi(\tilde{\gamma}) = \gamma$. From the definition of intrinsic length, $\Lambda_{\mathbb{H}}(\tilde{\gamma}) = \Lambda_{\mathbb{G}}(\gamma)$, and the equality of Carnot-Charathèodory distances follows immediately.

We are only left with the proof of the existence of the lift. Let $\gamma(s) = (u(s), v(s))$ be a horizontal curve in $\mathbb{G}$, $s \in [0,1]$. Define $\tilde{\gamma}(s) := [\tau(s), u(s), v(s)]$ and

$$\tau(s) = \tau_0 - \int_0^s \frac{\dot{v}(\xi)}{u(\xi)}\, d\xi, \tag{4}$$

with τ_0 arbitrary. Since γ is horizontal,

$$\Lambda_{\mathbb{G}}(\gamma) = \int_0^1 \left(\dot{u}(\xi)^2 + \left(\frac{\dot{v}(\xi)}{u(\xi)} \right)^2 \right)^{1/2} d\xi$$

is finite, hence the integral in (4) is well defined. Differentiating (4), and recalling (3), we obtain that

$$\dot{\tilde{\gamma}} = \dot{u}Y - \frac{\dot{v}}{u}X,$$

belongs to the horizontal space at $[\tau, u, v]$ almost everywhere. ■

The critical line of $\mathbb{G}$, $\{u = 0\}$, corresponds to the class of the cosets $\mathcal{S}P$ which lie in some horizontal plane. Indeed, a line $\tau \mapsto \mathcal{S}(\tau)P = (\tau, u, v + 1/2\tau u) = (x, y, t)$ is horizontal if $\frac{d\mathcal{S}(\tau)P}{d\tau}(\tau) \in H_{\mathcal{S}(\tau)P}\mathbb{H}$. On the other hand,

$$\frac{d\mathcal{S}(\tau)P}{d\tau}(\tau) = \partial_x + 1/2y\partial_t,$$

which is a horizontal vector exactly when $u = y = 0$. Hence, the union of the critical lines of the form $\mathcal{S}P$, $\mathcal{S}P$ being a critical point in $\mathbb{G}$, is the plane $y = 0$ in $\mathbb{H}$.

Corollary 2 *Let $(a, 0)$ and $(b, 0)$ be two points in $\mathbb{G}$, as in Problem B. Let Γ be the geodesic from $[0, a, 0]$ to $[A, b, 0]$ in $\mathbb{H}$. The solution of Problem B is the curve $\varphi \circ \pi(\Gamma)$.*

Theorem 3 *Consider now two points in $\mathbb{G}$, $\xi = (a, 0)$ and $\eta = (b, 0)$, $0 < a \le b$. Let $0 < A < +\infty$ and Γ be the geodesic from $[0, a, 0]$ to $[A, b, 0]$ in $\mathbb{H}$. Suppose that A is such that Γ does not intersect the plane $[\tau, 0, v]$*

Then, the solution of Problem A is the curve $\varphi \circ \pi(\Gamma)$.

Remark that there exist bounded open sets Ω in $\mathbb{G}$, such that $A = \int_\Omega \frac{dudv}{u^2} = +\infty$ and $\partial\Omega$ is a horizontal curve having finite $\mathbb{G}$-length. Indeed, let $a = b$. For any $\epsilon > 0$ we can find Ω for which $A = +\infty$ and $\Lambda_{\mathbb{G}}(\partial\Omega) = 2a + \epsilon$ and one cannot have $\epsilon = 0$. It suffices to consider the product of intervals $\Omega = [-\eta, a] \times [0, \eta^2]$, where η is the positive solution to $\eta^2 + a\eta - a\epsilon = 0$.

Remark 4 *Let $a = b = 1$ in Theorem 3. The assumptions of Theorem 3 are satisfied for $A \in (0, A_0]$, where A_0 is the unique positive number with the following property. There is a circle $c_0 = \partial D_0$ in the (x, y)-plane such that: (i) c_0 passes through $(0, 0)$ and $(0, A_0)$; (ii) c_0 is tangent to the line $y = -1$; (iii) the area of $D_0 \cap \{(x, y) : \ y < 0\}$ is A_0.*

The proof of Theorem 3 mainly consists in an application of the Gauss-Green formula. First, we need two lemmata.

Lemma 5 *Let Ω be a bounded open set in $\mathbb{G}$, having as boundary an horizontal curve $\gamma = \partial\Omega$ and let $\epsilon > 0$. Let $\Omega_\epsilon = \Omega \cap \{|u| > \epsilon\}$ and $l_\epsilon = \partial\Omega_\epsilon \setminus \partial\Omega$, which is the union of open intervals in the lines $\{u = \pm\epsilon\}$. If $\int_\Omega \frac{dudv}{u^2} < \infty$ then*

$$\lim_{\epsilon\to 0} \Lambda_{\mathbb{G}}(l_\epsilon) = 0. \tag{5}$$

Proof. First we recall that, by definition of horizontal curve, we have $\Lambda_{\mathbb{G}}(\partial\Omega) < \infty$. By the dominated convergence theorem,

$$\lim_{\epsilon\to 0} \Lambda_{\mathbb{G}}(\partial\Omega \cap \{|u| < \epsilon\}) = \Lambda_{\mathbb{G}}(\partial\Omega \cap \{|u| = 0\}) = 0.$$

The last equality depends on the fact that, for horizontal curves $\gamma = (u, v)$, we have $\Lambda_{\mathbb{G}}(\gamma) = \int \sqrt{\dot{u}^2 + \frac{\dot{v}^2}{u^2}} d\sigma$. Hence, it must be $\dot{\gamma} = 0$, a.e. on $\{u = 0\}$. The lemma is proved if

$$\Lambda_{\mathbb{G}}(l_\epsilon) \leq \Lambda_{\mathbb{G}}(\partial\Omega \cap \{|u| < \epsilon\}). \tag{6}$$

Consider $l_\epsilon^+ = l_\epsilon \cap \{u = \epsilon\}$ and $\Omega^+ = \Omega \cap \{u > 0\}$. For each $\eta \in l_\epsilon^+$, consider the half line s_η originating from η and moving to $u = 0$, parallel to the u-axis. Notice that $s_\eta \cap \partial\Omega \cap \{u > 0\} \neq \emptyset$, a.e. $\eta \in l_\epsilon^+$, since otherwise Ω^+ would be such that $\int_\Omega \frac{dudv}{u^2} = \infty$; let $F_\varepsilon^+ := \{\eta \in l_\epsilon^+ : \ s_\eta \cap \partial\Omega \cap \{u > 0\} \neq \emptyset\}$. Let

$$\Psi : \partial\Omega \cap \{0 < u < \epsilon\} \to l_\epsilon^+$$

The set $\partial\Omega \cap \{0 < u < \epsilon\} = \cup_j \partial\Omega_j$ is the countable union of open arcs $\gamma_j : (\alpha_j, \beta_j) \to \partial\Omega_j$, $j \in \mathbb{N}$. The map Ψ decreases the $\Lambda_{\mathbb{G}}$-length: $\sum_j \int_{\alpha_j}^{\beta_j} \sqrt{\dot{u}^2 + \frac{\dot{v}^2}{u^2}}\, d\sigma \geq \sum_j \int_{\alpha_j}^{\beta_j} \sqrt{\frac{\dot{v}^2}{\epsilon^2}}\, d\sigma \geq \Lambda_{\mathbb{G}}(l_\epsilon^+ - F)$; hence (6) holds. ■

Lemma 6 *Let Ω be a bounded open set in $\mathbb{G}$ having as (oriented) boundary the horizontal closed curve γ starting from $\xi = (a, 0)$ in $\mathbb{G} - \{u = 0\}$ and let $0 < \int_\Omega \frac{dudv}{u^2} < +\infty$. Then, the Gauss-Green formula holds on γ for the* 1*-form $\frac{dv}{u}$,*

$$\int_\gamma \frac{dv}{u} = -\int_\Omega \frac{dudv}{u^2}. \tag{7}$$

Proof. Let l_ϵ and Ω_ϵ as in lemma 5. Then,

$$\int_{\partial\Omega_\epsilon} \frac{dv}{u} = -\int_{\Omega_\epsilon} \frac{dudv}{u^2}.$$

The right hand side tends to $-\int_\Omega \frac{dudv}{u^2}$ as $\epsilon \to 0$. Equation (7) follows by the dominated convergence theorem applied to $\int_{\gamma\cap\{|u|>\epsilon\}} \frac{dv}{u}$. In fact,

$$\left|\int_{\gamma\cap\{|u|>\epsilon\}} \frac{dv}{u}\right| \leq \Lambda_{\mathbb{G}}(\gamma \cap \{|u| > \epsilon) \leq \Lambda_{\mathbb{G}}(\gamma) < \infty$$

and, from (5),

$$\Lambda_{\mathbb{G}}(l_\epsilon) = \left|\int_{l_\epsilon} \frac{dv}{u}\right| \to 0;$$

this proves the Lemma. ■

Proof of Theorem 3. We use the notation of Theorem 1. Consider points $E = [0, a, 0]$ and $F = [A, b, 0]$ in $\mathbb{H}$. Observe that $\xi = \varphi \circ \pi(E)$ and $\eta = \varphi \circ \pi(F)$. Since $A > 0$, the straight line joining E and F is not a vertical line in $\mathbb{H}$, hence there exists a *unique* minimizing geodesic Γ in $\mathbb{H}$ between E and F (see [Mont] and [Monti]). Let

$$\gamma_0 = \varphi \circ \pi(\Gamma)$$

be its projection onto $\mathbb{G}$. Since Γ does not intersect the plane $[\tau, 0, v]$, then γ_0 does not intersect the line $u = 0$ in $\mathbb{G}$, hence it is a horizontal curve from ξ to η.

Consider an open set Ω in $\mathbb{G}$ having as boundary the union of a horizontal curve γ_1 from ξ to η and the straight line returning from η

to ξ, and such that $A = \int_\Omega \frac{dudv}{u^2}$. Let Γ_1 be the horizontal lift of γ_1 to $\mathbb{H}$. By Lemma 6, Γ_1 joins E and F in $\mathbb{H}$:

$$\int_{\gamma_1} \frac{dv}{u} = \int_{\partial\Omega} \frac{dv}{u} = -\int_\Omega \frac{dudv}{u^2} = A.$$

Hence, $\Lambda_{\mathbb{H}}(\Gamma_1) \geq \Lambda_{\mathbb{H}}(\Gamma)$, i.e., $\Lambda_{\mathbb{G}}(\gamma_1) \geq \Lambda_{\mathbb{G}}(\gamma_0)$.

The curve γ_0, then, is the solution of Problem A provided it is the oriented boundary of an open set in $\mathbb{G}$. This amounts to showing that γ_0 does not have self-intersections, i.e. that $\varphi \circ \pi$ is injective on Γ. This is done in Lemma 7 below. ∎

Lemma 7 *Let Γ be the minimizing geodesic in $\mathbb{H}$ between $E = [0, a, 0]$ and $F = [A, b, 0]$. With the notation of Theorem 1, the map $\varphi \circ \pi$: $\mathbb{H} \to \mathbb{G}$ is injective on $\Gamma \cup \mathcal{L}'$, where $\mathcal{L}'$ is the straight line joining $[A, a, 0]$, $[A, b, 0]$.*

Proof. By symmetry we can always assume $b \geq a$. The problem is invariant under the intrinsic dilations of $\mathbb{G}$, $\delta_h(u, v) = (hu, h^2 v)$, hence we can consider the case $a = 1$, so that $\mathcal{L}'$ be the segment of horizontal straight line joining $[A, b, 0]$ and $[A, 1, 0]$ $\mathbb{H}$. Lemma 7 reduces to showing that the map $\varphi \circ \pi : \mathbb{H} \to \mathbb{G}$ is injective on $\Gamma \cup \mathcal{L}'$.

For $(x, y, t) \in \mathbb{H}$, let $L_{(x,y,t)}$ be the left translation in $\mathbb{H}$ by (x, y, t): $L_{(x,y,t)}(x_1, y_1, t_1) := (x, y, t) \cdot (x_1, y_1, t_1)$.

We change the Heisenberg coordinates back to the usual ones by (2) and we left-translate by $[0, 1, 0] = (0, 1, 0)$ to the origin.

We have to show that the map

$$f = \varphi \circ \pi \circ L_{(0,1,0)}$$

is injective on the curve $\Upsilon \cup \mathcal{L}$, where

$$\Upsilon = L_{(0,-1,0)}\Gamma, \ \mathcal{L} = L_{(0,-1,0)}\mathcal{L}'.$$

We denote by K the endpoint of Υ other than O, $K = (A, b-1, A\frac{b+1}{2})$.

If $(x, y, t) = L_{(0,-1,0)} \circ \Psi(\tau, u, v)$, then

$$(x, y, t) = \left(\tau, u - 1, v + \frac{\tau(u + 1)}{2}\right).$$

Observe that, if $f(P) = f(Q)$, then P and Q have the same y-coordinate. More precisely, $f(P) = f(Q)$ if and only if P and Q

belong to the same element from the sheaf of straight lines projecting $\mathbb{H}$ on $\mathbb{G}$. The lines of the sheaf have the form

$$\ell_{(u,v)} : \tau \mapsto \left(\tau, u-1, v + \frac{\tau(u+1)}{2}\right). \tag{8}$$

We have to consider two cases.

Case 1. The points $P \neq Q$ belong to Υ. They have the same y-coordinate and we assume that P is the one with smaller x-coordinate. We show that $f(P) = f(Q)$ leads to a contradiction.

Let Σ be the curve obtained by joining the arc of Υ from O to P, the segment $[P, Q]$ and, finally, the arc of Υ between Q and K. Consider now a new curve $\Xi = \Xi_1 \cup \Xi_2 \cup \Xi_3$, where the Ξ_j's are defined as follows. The curve Ξ_1 is the arc of Υ from O to P. The curve Ξ_2 is the horizontal straight line having speed $X(P) = \partial_x - \frac{y}{2}\partial_t = (1, 0, -\frac{y}{2})$, starting at Q and ending at $\tilde{Q}$, where Q and $\tilde{Q}$ belong to the same vertical line. The curve Ξ_3 is the vertical translation $\tilde{\Upsilon}$ of the geodesic Υ which starts at $\tilde{Q}$ and ends at $\tilde{K}$, the point of $\tilde{\Upsilon}$ lying on the vertical of K. A vertical translation of a curve in $\mathbb{H}$ is a curve obtained by a left translation of an element of the center Z.

Denote by c the circular arc obtained as the orthogonal projection of Υ onto the plane $t = 0$. Let D be region bounded by c and ℓ, the projection of $\mathcal{L}$ onto the $t = 0$ plane. Then, $\tilde{K} = (A, b-1, B)$, where B is the area of the portion of D lying above the line $y = u - 1$.

Since the difference of the t-coordinates between points in Υ and $\tilde{\Upsilon}$ is constant, then

$$t(Q) - t(\tilde{Q}) = A - B.$$

We have another way to compute this difference of areas. Let P' and Q' be, respectively, the vertical projections of P and Q onto the plane $t = 0$ and let p their Euclidean distance. We know the slope of the straight lines joining P with Q and $\tilde{Q}$, respectively, hence we can compute

$$t(Q) - t(\tilde{Q}) = \left(\frac{1+u}{2} - \frac{1-u}{2}\right) p = up,$$

which gives, together with the previous equality,

$$up = A - B. \tag{9}$$

We show now that this equation only holds in the trivial cases $p = 0$ or $u = 1$.

From now on, we work on the Euclidean plane $t = 0$ in $\mathbb{H}$. The u coordinate gives the signed distance from the line m having equation $y = u - 1$, to the line $y = -1$. The latter is mapped by f onto the critical line $u = 0$ in $\mathbb{G}$. The line m intercepts on c the chord $[P', Q']$, having length $p \geq 0$. Then, up is the signed area of the rectangle R having as side $[P', Q']$ and having the opposite side on the line $y = -1$. If $u < 0$, then, since $A - B \geq 0$, equation (9) is impossible unless $p = 0$. If $u \geq 0$, $A - B$ is the area between m and c.

We have two subcases, according to the segment $[P', Q']$ having y-coordinate smaller or greater than the y-coordinate of c's center. The former case is readily seen to be impossible, since the circular segment is strictly contained in the rectangle R.

Consider now the second case. Consider R_0, a rectangle like R, corresponding to the value $u = 1$ (i.e. one of the sides of R_0 has endpoints $(0, 0)$ and $(A, 0)$ and the area of R_0 is A). Consider now the following planar regions (see figure 1 below): $E_0 = D \cap \{(x, y) :\ 0 \leq x \leq A\}$; $E_1 = R_0 \setminus E_0$; $E_2 = D \setminus E_0$; $E_3 = R \setminus D$; E_4, the intersection of the half-plane $y \leq u - 1$ with $E_2 \setminus R$; $E_5 = R \cap D$; $E_6 = R_0 \cap D$. Finally, let A_j be the Euclidean area of E_j.

Since

$$A_1 + A_0 \geq A_1 + A_6 = A = A_2 + A_0$$
$$A_3 + A_5 = pu = A - B = A_4 + A_5,$$

we have $A_1 \geq A_2$, $A_3 = A_4$. On the other hand, by inclusion,

$$A_1 < A_3 \quad \text{and} \quad A_4 < A_2,$$

which implies $A_1 < A_2$. Contradiction. Hence, $pu > A - B$.

Case 2. Suppose now that $P \in \upsilon$, $Q \in \mathcal{L}$ and let P', Q' be their projections onto the $t = 0$ plane, respectively. P' and Q' have the same y-coordinate and the x-coordinate of P' is larger that that of Q'. Let D be as in Case 1. The area of D is A. Let B be the area of the portion of D lying below the line $[OP']$, E be the area of the portion of D lying between the line $[OP']$ and $[P'Q']$ and let $C = A - B - E$ be the area of the remaining part of D. Let Ξ_1 and Ξ_2 be defined as in Case 1. $\Xi_1 \cup \Xi_2$ is horizontal, hence the t-coordinate of its endpoint $\tilde{Q}$ is $B + E$. On the other hand, the t-coordinate of Q is $A = B + E + C$ and that of P is B.

Let $p > 0$ be the Euclidean distance between Q' and P'. Similarly to Case 1, we compute

$$0 < C = A-(B+E) = t(Q)-t(\tilde{Q}) = \left(\frac{1+u}{2} - \frac{1-u}{2}\right)(-p) = -up < 0,$$

absurd. ■

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