# Basic inequalities

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A function $\Phi : I \to \mathbb{R}$ defined on an interval $I \subseteq \mathbb{R}$ is *convex* if for all integer $n \geq 2$:

$$\sum_{j=1}^{n} t_j = 1,\ t_1, \dots, t_n \geq 0,\ a_1, \dots, a_n \in I \implies \Phi\left(\sum_{j=1}^{n} t_j a_j\right) \leq \sum_{j=1}^{n} t_j \Phi(a_j).$$

By induction, $\Phi$ is convex iff the inequality above holds for $n = 2$.
**Jensen's inequality.**[1] Let $\Phi : [0, +\infty) \to [0, \infty)$ be a convex function and let $(X, \mu)$ be a probability measure space. If $f \geq 0$ is a measurable function on $X$, then

$$\Phi\left(\int_X f d\mu\right) \leq \int_X \Phi(f) d\mu. \tag{1}$$

**Proof.** Let $f = \sum_j a_j \chi_{E_j}$ be a simple function: $\{E_j\}$ is a countable, measurable partition of $X$. Then, by convexity,

$$\begin{aligned} \Phi\left(\int_X f d\mu\right) &= \Phi\left(\sum_{j=1}^{n} a_j \mu(E_j)\right) \\ &\leq \sum_{j=1}^{n} \Phi(a_j)\, \mu(E_j) = \int_X \Phi(f) d\mu. \end{aligned}$$

For general $f \geq 0$, let $\{f_n\}$ be a sequence of simple functions such that $f_n \nearrow f$. The desired inequality follows by a simple limiting argument[2]. ■

We can dispense with the positivity assumption provided $f$ is integrable.

**Proposition 1** *Let $\Phi : (a, b) \to \mathbb{R}$ be convex, $-\infty \leq a < b \leq +\infty$, and let $(X, \mu)$ be a probability space. If $f : X \to \mathbb{R}$ is integrable and $f(X) \subseteq (a, b)$, then*

$$\Phi\left(\int_X f d\mu\right) \leq \int_X \Phi(f) d\mu.$$

**Proof.** Let $a < u < w < v < b$. By convexity,

$$\begin{aligned} w &= \frac{v - w}{v - u} u + \frac{w - u}{v - u} v \implies \\ \Phi(w) &\leq \frac{v - w}{v - u} \Phi(u) + \frac{w - u}{v - u} \Phi(v) \implies \end{aligned}$$

[1] Some words on extremals?

[2] It is useful to split $\Phi = \Phi_1 + \Phi_2$, with $\Phi_1$ increasing and $\Phi_2$ decreasing. Use Monotone Convergence with $\Phi_1$ and Dominated Convergence with $\Phi_2$

$$\begin{aligned}(\Phi(w)-\Phi(u))(v-w) &\leq (\Phi(v)-\Phi(w))(w-u) \implies \\ \frac{\Phi(w)-\Phi(u)}{w-u} &\leq \frac{\Phi(v)-\Phi(w)}{v-w}.\end{aligned}$$

[3] Then, there is $C(w)\in\mathbb{R}$ such that

$$\Phi(t)\geq\Phi(w)+C(w)(t-w)$$

whenever $t\in(a,b)$.

Let now $w=\int_X fd\mu\in(a,b)$, by the Mean Value Theorem, and let $t=f(x)$. Integrating w.r.t. $\mu$,

$$\begin{aligned}\int_X \Phi(f(x))d\mu(x) &\geq \Phi\left(\int_X fd\mu\right)+C\int_X\left(f(x)-\int_X fd\mu\right)d\mu(x) \\ &= \Phi\left(\int_X fd\mu\right).\end{aligned}$$

■ [4]

**Exercise 4** *Suppose that $\Phi$ is also increasing and that for all*[5] *$T>0$ there is*

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[3] Observe that the inequality in the second line also gives

$$\frac{\Phi(v)-\Phi(u)}{v-u}\leq\frac{\Phi(v)-\Phi(w)}{v-w}.$$

[4] **A different proof.**

**Lemma 2 (An extension of the Monotone Convergence Theorem.)** *Suppose that $\varphi_n\in L^(\mu)$ for $n\geq1$ and that $\varphi_n\nearrow\varphi$. Then,*

$$\int\varphi_n d\mu\nearrow\int\varphi d\mu.$$

**Proof.** Let $\psi_n=\varphi_n\vee0\nearrow\psi=\varphi\vee0$ and $\eta_n=\varphi_n\wedge0\nearrow\eta=\varphi\wedge0$. Use MCT for $\psi_n$ and DCT for $\eta_n$. ■

**Lemma 3** *Let $\Phi:(a,b)\to\mathbb{R}$ be a convex function and let $a<\alpha<\beta<b$. Then, there exist*

$$-\infty<\Phi'(\alpha+)\leq\Phi'(\beta-)<+\infty.$$

**Proof.** Whenever $0<h,k<\frac{\alpha+\beta}{2}$, we have

$$\frac{\Phi(\alpha+h)-\Phi(\alpha)}{h}\leq\frac{\Phi(\beta)-\Phi(\beta-k)}{k}.$$

The LHS decreases as $h\to0$, while the RHS increases as $k\to0$. Observe that both RHS and LHS are bounded. Take limits. ■

**Proof.** of Proposition 1 For $a<\alpha<\beta<b$, let

$$\Phi_\alpha^\beta(t)=\begin{cases}\Phi(\beta)+\Phi'(\alpha+)(t-\alpha) \text{ if } t\in(a,\alpha] \\ \Phi(t) \text{ if } t\in[\alpha,\beta] \\ \Phi(\beta)+\Phi'(\beta-)(t-\beta) \text{ if } t\in[\beta,b).\end{cases}$$

Then, $\Phi_\alpha^\beta$ is convex by the second lemma, $\Phi_\alpha^\beta\leq\Phi$ and, if $\alpha_n\searrow a$ and $\beta_n\nearrow b$, then $\Phi_{\alpha_n}^{\beta_n}\nearrow\Phi$.

If $f\in L^1(\mu)$, then $\varphi_n=\Phi_{\alpha_n}^{\beta_n}\circ f\in L^1(\mu)$ and $\varphi_n\nearrow\varphi=\Phi\circ f$. By the first lemma, the inequality is reduced to

$$\Phi_{\alpha_n}^{\beta_n}\left(\int fd\mu\right)\leq\int\Phi_{\alpha_n}^{\beta_n}(f)\,d\mu.$$

This last inequality can be proved similarly to (1): for simple $f$ it reduces to the definition of convex function; for $f\in L^1$ use DCT. ■

[5] Or, which is the same, for just one such $T$.

*$C > 0$ such that*

$$\Phi(Tx) \le C\Phi(x). \tag{2}$$

*($\Phi(t) = t^p,\ p \ge 1$ is a function with these properties).*

*Show that, if we replace the assumption that $\mu(X) = 1$ by $\mu(X) < \infty$, we obtain the inequality*

$$\Phi\left(\int_X f d\mu\right) \le C(\mu(X)) \int_X \Phi(f) d\mu. \tag{3}$$

*Find an example of a convex, increasing function $\Phi$ such that (2) and (3) both fail.*

**Exercise 5** *Let $\psi : [0, \infty) \to [0, \infty)$, $\psi(0) = 0$, $\psi(x) = x \log(1/x)$ if $x \ne 0$. Let $P = \{p_j\}_{1 \le j \le n}$ be a probability distribution: $\sum_{j=1}^n p_j = 1$, $p_j \ge 0$. The* entropy *of $P$ is $\mathcal{E}(P) = \sum_{j=1}^n \psi(p_j)$. Prove that the estimates*

$$0 \le \mathcal{E}(P) \le \log n$$

*hold and that they are sharp. What are the extremals?*

**Hölder's inequality.** If $f, g \ge 0$ are nonnegative and measurable on the measure space $(Y, dx)$, $1 \le p \le \infty$ and $p'$ is the exponent conjugate to $p$, $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\int fg dx \le \|f\|_{L^p} \|g\|_{L^{p'}}.$$

**Proof.** The case $p = \infty$ or $p = 1$ is elementary, so we assume $1 < p < \infty$. We use the convexity of $t \to t^p$ and Jensen's inequality with the measure space $(Z, \mu)$, where $Z$ is the support of $g$ and $d\mu = \frac{g^{p'}}{\|g\|^{p'}_{L^{p'}}} dx$.

$$\begin{aligned}
\int fg dx &= \int f g^{1-p'} \frac{g^{p'}}{\|g\|^{p'}_{L^{p'}}} dx \cdot \|g\|^{p'}_{L^{p'}} \\
&\le \|g\|^{p'}_{L^{p'}} \left[ \int \left( f g^{1-p'} \right)^p \frac{g^{p'}}{\|g\|^{p'}_{L^{p'}}} dx \right]^{1/p} \\
&\le \|f\|_{L^p} \|g\|^{p' - \frac{p'}{p}}_{L^{p'}} = \|f\|_{L^p} \|g\|_{L^{p'}}.
\end{aligned}$$

■

We have equality in Hölder's inequality if and only if $g^{p'} = f^p$ *a.e.*.

**Iterated Hölder's inequality.** If $p_j \in [1, \infty]$, $\sum_j \frac{1}{p_j} = 1$ and $f_j \ge 0$ is a family of measurable functions, then

$$\int \Pi_j f_j dx \le \Pi_j \|f_j\|_{L^{p_j}}.$$

The inequality follows from two-Hölder's by induction.

There is a continuous generalization of Hölder's inequality, which can be stated as follows. Let $\mu$ be a probability measure on some space $X$ and $h = h(t, x) : X \times Y \to \mathbb{R}$ be mesurable and nonnegative. Then,

$$\log \left[ \int_Y \exp\left( \int_X h(t,x) d\mu(t) \right) dx \right] \le \int_X \log \left[ \int_Y \exp(h(t,x)) dx \right] d\mu(t).$$

This inequality follows easily from iterated Hölder's and an approximation argument.[6]

Let $f, g$ be nonnegative, measurable functions on $\mathbb{R}$. The *convolution* of $f$ and $g$ is $f * g : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$:

$$f * g(x) = \int f(x-y)g(y)dy.$$

Note that the convolution can be defined as well among sequences with indeces in $\mathbb{Z}$:

$$(a * b)_n = \sum_{m \in \mathbb{Z}} a_{n-m} b_m,$$

and functions defined on the circle:

$$f * g(e^{i\alpha}) = \int_{-\pi}^{\pi} f(e^{i(\alpha-\theta)})g(e^{i\theta})\frac{d\theta}{2\pi}.$$

In general, it makes sense to define convolutions whenever we have a group with a (left) translation invariant measure.

**Young's inequality.** Suppose that $f, g$ are nonnegative and measurable on $\mathbb{R}$ and that $p, q, r \in [1, +\infty]$ are such that

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \geq 0. \tag{4}$$

(The conditions imply that $r \geq p, q$.) Then,

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}. \tag{5}$$

**Proof.** The case $r = +\infty$ is contained in Hölder's inequality. Consider $r \in (1, \infty)$ first. Then $p, q \in [1, \infty)$. Let $P, Q \in [1, \infty)$, to be chosen below, be such that $P^{-1} + Q^{-1} + r^{-1} = 1$ and let $a, b \in [0, 1]$ be such that

$$p = aP = (1-a)r,\ q = bQ = (1-b)r,\ i.e.\ a = \frac{r}{P+r} = \frac{r}{\frac{p}{a}+r},\ b = \frac{r}{Q+r} = \dots.$$

Hence,

$$a = \frac{r-p}{r},\ b = \frac{r-q}{r}.$$

Let's verify the condition on $P, Q$:

$$\frac{1}{P} + \frac{1}{Q} = \frac{a}{p} + \frac{b}{q} = \frac{1}{p} - \frac{1}{r} + \frac{1}{q} - \frac{1}{r} = 1 - \frac{1}{r},$$

as wished. By Hölder's inequality,

$$\begin{aligned} f * g(x) &= \int f(x-y)g(y)dy = \int f(x-y)^a g(y)^b f(x-y)^{1-a} g(y)^{1-b} dy \\ &\leq \left[\int (f(x-y)^{aP} dy\right]^{1/P} \left[\int (g(y)^{bQ} dy\right]^{1/Q} \\ &\cdot \left[\int f(x-y)^{(1-a)r} (g(y)^{(1-b)r} dy\right]^{1/r}. \end{aligned}$$

[6] Is there a direct proof?

Taking into account the fact that the Lebesgue measure is translation invariant and the relations on $a, b, P, Q, p, q$, we have

$$\begin{aligned}\int [f(x-y)g(y)]^r dx &\leq \|f\|_{L^p}^{pr/P} \|g\|_{L^q}^{qr/Q} \int \left[ \int f(x-y)^p g(y)^q dy \right] dx \\ &= \|f\|_{L^p}^{p(1+r/P)} \|g\|_{L^q}^{q(1+r/Q)} = \|f\|_{L^p}^r \|g\|_{L^q}^r.\end{aligned}$$

■

**Exercise 6** *Prove that Young's inequality (5) in* $\mathbb{R}$ *holds in the cases (4) only.* **Suggestion:** *let* $\delta_\lambda f(x) = f(x/\lambda)$, $\lambda > 0$*; insert* $\delta_\lambda f$, $\delta_\lambda g$ *instead of* $f$ *and* $g$ *in (5) and let* $\lambda$ *renge in* $(0, \infty)$.

We have not proved the important (easier) case $p = q = r = 1$. For these values of the exponents, Young's inequality says that $(L^1, *)$ is a *Banach algebra*.

**Exercise 7** *Prove the case* $p = q = r = 1$ *of Young's inequality.*

**Problem.** Find an iterated version for Young's inequality and, if there is one, write down a continuos version of it.

The best constant in Young's inequality in $\mathbb{R}^n$ was found by W. Beckner in 1975 [Beck].

**Schur' Lemma.** Let $X$ be a measure space and let $K : X \times X \to \mathbb{R}$ be a nonnegative, measurable functions. Define an operator $T$ defined by the kernel $K$. If $f$ is a nonnegative, measurable function, then

$$Tf(x) = \int K(x,y)f(y)dy.$$

Let $1 < p < \infty$. Suppose that there is a strictly positive function $\lambda$ on $X$ such that

$$\int K(x,y)\lambda^{p'}(y)dy \leq C\lambda^{p'}(x), \quad \int K(x,y)\lambda^p(x)dx \leq C\lambda^p(y).$$

Then, $T$ is bounded from $L^p$ to $L^p$.

**Proof.** Using Hölder's from first to second line with measure $K(x,y)dy$ and the hypothesis

$$\begin{aligned}\int (Tf)^p(x)dx &= \int \left( \int K(x,y)\lambda(y)\lambda^{-1}(y)f(y)dy \right)^p dx \\ &\leq \int \left( \int K(x,y)\lambda^{p'}(y)dy \right)^{p/p'} \left( \int K(x,y)\lambda^{-p}(y)f(y)^p dy \right) dx \\ &\leq C \int \lambda^p(x) \left( \int K(x,y)\lambda^{-p}(y)f(y)^p dy \right) dx \\ &= C \int \lambda^{-p}(y)f(y)^p \left( \int K(x,y)\lambda^p(x)dx \right) dy \\ &\leq C' \int f(y)^p \lambda^{-p}(y)\lambda^p(y)dy = C' \int f(y)^p dy.\end{aligned}$$

■

**Exercise 8** *Find a Schur's Lemma ensuring that* $T : L^p \to L^q$, $1 < q \leq p < \infty$.

Jensen's inequality deals with a probability space. A consequence of the inequality is that $L^q(\mu) \subset L^p(\mu)$ if $\mu$ is a probability measure and $p < q$. At the opposite end, we have the measure space $\mathbb{N}$ with the counting measure. Here, $\ell^1 \subset \ell^\infty$.

**Proposition 9** *Let $a = \{a_k\}_{k\geq 0}$ be a sequence of nonnegative numbers. If $p < q$, then $\|a\|_q \leq \|a\|_p$.*

**Proof.** Since $t^{q/p} \leq t$ when $t \in [0,1]$,

$$\frac{\sum a_k^q}{(\sum a_h^p)^{q/p}} = \sum \left(\frac{a_k^p}{\sum a_h^p}\right)^{q/p} \leq \sum \left(\frac{a_k^p}{\sum a_h^p}\right) = 1.$$

■

**Exercise 10** *Prove the following. Let $\Phi : [0,\infty) \to [0,\infty)$ be such that $\frac{\Phi(x)}{x}$ is increasing. Then,*

$$\sum \Phi(a_k) \leq \Phi(\sum a_k)$$

*if $a_k \geq 0$.*

*For instance,*

$$\sum (e^{a_k} - 1) \leq e^{\sum a_k} - 1.$$

Note that here, too, a differential inequality is the key to a class of integral inequalities.

An other application of convexity is the proof of *Minkowsky's inequality.* Let $\Phi : [0,\infty) \to [0,\infty)$ be a convex, increasing function such that $\Phi(0) = 0$. For a measurable function $f$ (on some fixed measure space), let

$$\|f\|_\Phi = \inf\left\{C > 0 : \int \Phi\left(\frac{|f|}{C}\right) dx \leq 1\right\}.$$

Here, $\inf \emptyset = +\infty$, by definition.

**Exercise 11** *If $\Phi(t) = t^p$, $p \geq 1$, then $\|f\|_\Phi = \|f\|_{L^p}$.*

**Theorem 12**

$$\|f + g\|_\Phi \leq \|f\|_\Phi + \|g\|_\Phi.$$

**Proof.** Let $a, b > 0$ be s.t. $\int \Phi\left(\frac{|f|}{a}\right) dx \leq 1$, $\int \Phi\left(\frac{|g|}{b}\right) dx \leq 1$. By convexity,

$$\begin{aligned}\int \Phi\left(\frac{|f+g|}{a+b}\right) dx &\leq \int \Phi\left(\frac{a}{a+b}\frac{|f|}{a} + \frac{b}{a+b}\frac{|g|}{b}\right) dx \\ &\leq \int \frac{a}{a+b}\Phi\left(\frac{|f|}{a}\right) + \frac{b}{a+b}\Phi\left(\frac{|g|}{b}\right) dx \\ &\leq 1. \end{aligned} \tag{6}$$

Hence, $a + b \geq \|f + g\|_\Phi$. The thesis follows by passing to infima. ■

Under suitable hypothesis, Minkowsky's inequality has the following integral generalization:

$$\left\|\int_X f(t,\cdot)d\lambda(t)\right\|_\Phi \leq \int_X \|f(t,\cdot)\|_\Phi d\lambda(t).$$

The classic of inequalities is [HLP].

## References

[Beck] W. Beckner *Inequalities in Fourier analysis.* Ann. of Math. (2) **102** (1975), no. 1, 159–182.

[HLP] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities.*Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1988. xii+324 pp.