(1) Siano $\Omega = \{(x,y,z) \in \mathbb{R}^3 : \ xy \le z \le xy+1;\ 0 \le x \le 1;\ 0 \le y \le 1\}$
e $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, $F(x,y,z) = (x^2, y^2, z^2)$.

(1.1) Scrivere e calcolare il flusso di F attraverso $\partial\Omega$ orientato secondo la normale esterna (flusso uscente).

(1.2) Calcolare il flusso di F attraverso Σ, dove

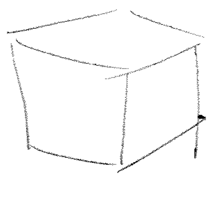
$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : \ z = xy;\ 0 \le x \le 1;\ 0 \le y \le 1\},$$

orientata secondo μ, dove $\mu(0,0,0) = (-1,0,0)$.

Svolgimento. Posso calcolare il flusso di F attraverso $\partial\Omega$ usando il Teorema di Gauss.

$\operatorname{div} F(x,y,z) = 2x + 2y + 2z$; $\operatorname{div} F : \mathbb{R}^3 \to \mathbb{R}$ e

$$I = \iint_{\partial\Omega} F \cdot \nu \, d\sigma = \iiint_{\Omega} (2x + 2y + 2z) \, dx\, dy\, dz.$$



Integro "per fili". Sia $Q = [0,1] \times [0,1]$
$= \{(x,y) \in \mathbb{R}^2 : \ 0 \le x \le 1 \text{ e } 0 \le y \le 1\}$.

Allora $\Omega = \{(x,y,z) : \ (x,y) \in A \text{ e } xy \le z \le xy+1\}$.

Quindi, se $f \in C(\overline{\Omega}, \mathbb{R})$, $\displaystyle\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iint_A \left\{ \int_{xy}^{xy+1} f(x,y,z)\,dz \right\} dx\,dy$

In particolare:

$$\iiint_{\Omega} (2x+2y+2z)\,dx\,dy\,dz = \iint_{[0,1]\times[0,1]} dx\,dy \left\{ \int_{xy}^{xy+1} (2x+2y+2z)\,dz \right\}$$

$$= \int_0^1\!\!\int_0^1 dx\,dy \left\{ 2x \cdot 1 + 2y \cdot 1 + (xy+1)^2 - (xy)^2 \right\}$$

$$= \int_0^1\!\!\int_0^1 dx\,dy\,(2x + 2y + 2xy + 1) = 1 + 1 + \frac{1}{2} + 1 = \frac{7}{2};$$

$$\boxed{\iint_{\partial\Omega} F \cdot \nu \, d\sigma = \frac{7}{2}}$$

Per scrivere il flusso di F attraverso $\partial\Omega$ parametrizzo $\partial\Omega$, dopo averlo diviso in $\underline{\underline{6}}$ facce regolari.

(c+) $\Sigma_{c+} = \{(x,y,z) : z = xy+1 ; 0 \le x \le 1 ; 0 \le y \le 1\} = \Phi_{c+}(Q)$

dove $\Phi_{c+}(x,y) = (x, y, xy+1)$; $\Phi_{c+} : Q \to \mathbb{R}^3$

$$\partial_x \Phi_{c+} \times \partial_y \Phi_{c+} = (1,0,y) \times (0,1,x) = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1)$$

(c−) $\Sigma_{c-} = \{(x,y,z) : z = xy ; (x,y) \in Q\} = \Phi_{c-}(Q)$ con

$\Phi_{c-}(x,y) = (x, y, xy)$; $\Phi_{c-} : Q \to \mathbb{R}^3$

$$\partial_x \Phi_{c-}(x,y) \times \partial_y \Phi_{c-}(x,y) = (-y, -x, 1)$$

(a−) $\Sigma_{a-} = \{(x,y,z) : x = 0 ; 0 \le y \le 1 ; 0 = 0 \cdot y \le z \le 0 \cdot y + 1 = 1\} = \Phi_{a-}(A_-)$

dove $\Phi_{a-}(y,z) = (0; y; z)$; $\Phi_{a-} : A_- = \{(y,z) : 0 \le y \le 1 ; 0 \le z \le 1\} \to \mathbb{R}^3$

$$\partial_y \Phi_{a-}(y,z) \times \partial_z \Phi_{a-}(y,z) = (0;1;0) \times (0;0;1) = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1, 0, 0)$$

(a+) $\Sigma_{a+} = \{(x,y,z) : x = 1 ; 0 \le y \le 1 ; y = 1 \cdot y \le z \le 1 \cdot y + 1 = y+1\} = \Phi_{a+}(A+)$

dove $\Phi_{a+}(y,z) = (1; y; z)$; $\Phi_{a+} : A+ = \{(y,z) : 0 \le y \le 1 ; y \le z \le y+1\} \longrightarrow \mathbb{R}^3$

e $\partial_y \Phi_{a+}(y,z) \times \partial_z \Phi_{a+}(y,z) = (0,1,0) \times (0,0,1) = (1,0,0)$

(b−) $\Sigma_{b-} = \{(x,y,z) : y = 0 ; 0 \le x \le 1 ; 0 \le z \le 1\} = \Phi_{b-}(B_-)$

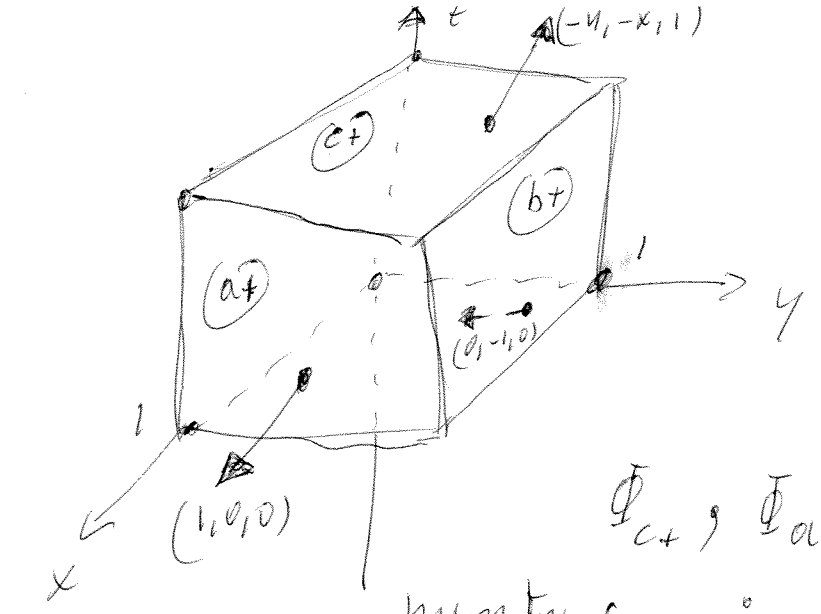
dove $\Phi_{b-}(x,z) = (x, 0, z)$; $\Phi_{b-} : B_- = \{(x,z) : 0 \le x \le 1 ; 0 \le z \le 1\} \to \mathbb{R}^3$

$$\partial_x \Phi_{b-} \times \partial_z \Phi_{b-} = (1,0,0) \times (0,0,1) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0,-1,0)$$

(b+) $\Sigma_{b+} = \{(x,y,z) : y = 1 ; 0 \le x \le 1 ; x \le z \le x+1\} = \Phi_{b+}(B+)$

dove $\Phi_{b+}(x,z) = (x; 1, z)$; $\Phi_{b+} : B_+ = \{(x,z) : 0 \le x \le 1 ; x \le z \le x+1\} \to \mathbb{R}^3$

e $\partial_x \Phi_{b+} \times \partial_z \Phi_{b+} = (1,0,0) \times (0,0,1) = (0,-1,0)$



Sono <u>compatibili</u> con le normali <u>esterne</u> a $\partial\Omega$ le parametrizzazioni:

$$\Phi_{c+},\ \Phi_{a+},\ \Phi_{b-}$$

mentre sono <u>incompatibili</u> $\Phi_{c-},\ \Phi_{a-},\ \Phi_{b+}$.

Quinoli:

$$\iint_{\partial\Omega} (F\cdot\nu)\, d\sigma =$$

$$+\int_0^1 dx \int_0^1 dy \ \left(x^2, y^2, (xy+1)^2\right)\cdot(-y,-x,1) \ \cancel{dx\,dy} \qquad (c+)$$

$$-\int_0^1 dx \int_0^1 dy \ \left(x^2, y^2, (xy)^2\right)\cdot(-y,-x,1) \ \cancel{dx\,dy} \qquad (c-)$$

$$-\int_0^1 dy \int_0^1 dz \ \left(0, y^2, z^2\right)\cdot(1,0,0) \ \cancel{dy\,dz} \qquad (a-)$$

$$+\int_0^1 dy \int_y^{y+1} dz \ \left(1, y^2, z^2\right)\cdot(1,0,0) \ \cancel{dy\,dz} \qquad (a+)$$

$$+\int_0^1 dx \int_0^1 dz \ \left(x^2, 0, z^2\right)\cdot(0,-1,0) \qquad (b-)$$

$$-\int_0^1 dx \int_x^{x+1} dz \ \left(x^2, 1, z^2\right)\cdot(0,-1,0) \qquad (b+)$$

Poiché $(\Sigma, \mu) = (\Sigma_{c-}, \nu)$ e $\operatorname{Rot} F = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 2x & 2y & 2z \end{vmatrix} = 0$,

$$\boxed{\iint_{\Sigma} (\operatorname{Rot} F)\cdot\mu \, d\sigma = 0.}$$

② Trovare l'integrale generale di

$$(E)\quad x'' - 2x' + x = 3e^t + 5\cdot\sin t + 7\cdot e^t\cdot\cos(t).$$

Svolgimento. Trovo quello dell'omogenea associata:

$$(E0)\quad y'' - 2y' + y = 0$$

Risolvo: $\lambda^2 - 2\lambda + 1 = 0$ $\quad \lambda_1 = 1 = \lambda_2$: "risonanza"

$$y(t) = A\cdot e^t + B\cdot t\cdot e^t$$

Poiché sia e^t che $t\cdot e^t$ risolvono $(E0)$, il termine $3\cdot e^t$ dà ulteriore risonanza.
Gli addendi $5\sin(t)$ e $7\cdot e^t\cdot\cos(t)$ non danno invece problemi. Provo con

$$x(t) = at^2\cdot e^t + b\cdot\cos t + c\cdot\sin t + p\cdot e^t\cos t + q\cdot e^t\sin t$$

$$x'(t) = 2at\cdot e^t + at^2\cdot e^t - b\sin t + c\cdot\cos t + e^t\cos t(p+q) + e^t\sin t(q-p)$$

$$x''(t) = 2a e^t + 4at\, e^t + at^2 e^t - b\cos t - c\cdot\sin t + e^t\cos t(p+q+q-p) + e^t\sin t(q-p-p-q)$$

Sostituendo:

$$3e^t + 5\cdot\sin t + 7\cdot e^t\cdot\cos(t) =$$

$$2ae^t + 4ate^t + at^2e^t - b\cos t - c\sin t + 2q\,e^t\cos t - 2p\cdot e^t\sin t$$

$$- 4ate^t - 2at^2e^t + 2b\sin t - 2c\cos t - 2e^t\cos t(p+q) - 2e^t\sin t(q-p)$$

$$+ at^2e^t + b\cos t + c\sin t + p\,e^t\cos t + q\,e^t\sin t$$

$$= 2a\cdot e^t + (-2c)\cdot\cos t + (2b)\cdot\sin t + (-p)\cdot e^t\cos t + (-q)\cdot e^t\sin t$$

$$\Rightarrow\quad a = 3/2;\quad c = 0;\quad b = 5/2;\quad p = -7;\quad q = 0;$$

$$\boxed{x(t) = Ae^t + Bt\cdot e^t + \tfrac{3}{2}t^2\cdot e^t + \tfrac{5}{2}\cdot\cos t - 7\cdot e^t\cdot\cos t}$$