FUNCTION SPACES RELATED TO THE DIRICHLET SPACE

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We report on recent joint work with R. Rochberg, E. Sawyer and B. Wick, related with the holomorphic Dirichlet space and we contextualize it within the general theory.

1. An old and prestigious story: the Hardy space.

Consider the Hardy space H^2 in the unit disc $\mathbb{D}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \implies \|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

The *multiplier space* of H^2, containing all g's holomorphic in $\mathbb{D}$ such that the operator $f \mapsto gf$ is bounded on H^2, is $\mathcal{M}(H^2) = H^\infty$, the space of the bounded holomorphic functions. We also have that

$$H^2 \cdot H^2 := \left\{h = fg : \ f, g \in H^2\right\} = H^1 \hookleftarrow H^2$$

is the product space of H^2, by inner/outer factorization and Cauchy-Schwarz inequality. It is interesting, then, to find the dual space of H^1. C. Fefferman [7] proved that, under the H^2 paring (with some care), $(H^2 \cdot H^2)^* = (H^1)^* = BMO \cap H(\mathbb{D})$ is the space of the analytic functions with bounded mean oscillation. The definition of BMO, born out of a problem in elasticity theory [9], in our context is as follows. A complex valued function b on the torus $\mathbb{T}$ has *bounded mean oscillation* if there is a positive constant C such that

$$\frac{1}{|I|}\int_I \left|f(e^{i\theta}) - \frac{1}{|I|}\int_I f(e^{i\psi})d\psi\right|^2 d\theta \le C$$

for all subarcs I of $\mathbb{T}$. The BMO norm of f is the best C we can put in the inequality (assume $f(0) = 0$ to make it truly a norm). A different characterization of the BMO norm for analytic function was used in establishing this and other results. Let μ be a positive measure on the unit disc. The *Carleson measure norm* of b is

$$[\mu]_{CM(H^2)} := \sup_{f\neq 0} \frac{\int_{\mathbb{D}} |f|^2 d\mu}{\|f\|_{H^2}^2} \approx \sup_I \frac{\mu(S(I))}{|I|}.$$

In the term on the far right, $S(I) = \{z \in \mathbb{D} : \ z/|z| \in I, \ 1 - |z| < |I|2\pi\}$ is the Carleson box based on the arc I, and the equivalence $\approx$ is Carleson's characterization of μ's in $CM(H^2)$ [4]. Let b an analytic function on $\mathbb{D}$ and let $d\mu_b = (1-|z|^2)|b'|^2 dA$ (dA is area measure on $\mathbb{D}$). Then, $\|b\|_{BMO} \approx [\mu_b]_{CM(H^2)}^{1/2}$ (the measure μ_b is an important object in H^2: $\|f\|_{H^2}^2 \approx \int_{\mathbb{D}} d\mu_f$). We have then a sequence of Banach spaces naturally arising in the Hilbertian theory of H^2:

$$H^\infty = \mathcal{M}(H^2) \hookrightarrow BMOA = (H^2 \cdot H^2)^* \hookrightarrow H^2 \hookrightarrow H^1 = H^2 \cdot H^2.$$

The story we are telling has a chapter concerning bilinear forms. Given b analytic in $\mathbb{D}$, let $T_b^{H^2} : H^2 \times H^2 \to \mathbb{R}$ be the bilinear Hankel form $T_b^{H^2}(f,g) =< fg, b >_{H^2}$. Nehari [12] proved that

$$\|T_b^{H^2}\|_{H^2\times H^2} := \sup \frac{|T_b^{H^2}(f,g)|}{\|f\|_{H^2}\|g\|_{H^2}} = \|b\|_{(H^2 \cdot H^2)^*} \approx \|b\|_{BMO} \approx [\mu_b]_{CM(H^2)}^{1/2}$$

(the two $\approx$'s are Fefferman's fundamental conribution to the theory).

2. A developing story: the Dirichlet space.

Consider the Dirichlet space $\mathcal{D}$, containing the functions f holomorphic in $\mathbb{D}$ for which the seminorm

$$\|f\|_{\mathcal{D}} = \left(\int_{\mathbb{D}} |f'(z)|^2\right)^{1/2}$$

is finite. We assume throughout that $f(0) = 0$, so to make $\|f\|_{\mathcal{D}}$ into a norm. The multiplier space $\mathcal{M}(\mathcal{D})$ of $\mathcal{D}$ contains the functions g such that $f \mapsto gf$ is bounded on $\mathcal{D}$, and it is easily seen that it consists of those bounded functions g for which the measure $d\mu = d\mu_g = |g'|^2 dA$ satisfies

$$[\mu]_{CM(\mathcal{D})} := \sup_{f\neq 0} \frac{\int_{\mathbb{D}} |f|^2 d\mu}{\|f\|_{\mathcal{D}}^2} < +\infty.$$

Measures (not necessarily arising from a function g) with this imbedding properties are called *Carleson measures for* $\mathcal{D}$, and they were characterized by Stegenga [13] in terms of a capacitary condition. Let $E = \cup_j I_j$ the disjoint union of closed subarcs of the unit circle and let $S(E) = \cup_j S(I_j)$ be the union of the corresponding Carleson boxes. The Carleson measure norm in $\mathcal{D}$ of a positive measure μ is

$$[\mu]_{CM(\mathcal{D})} \approx \mu(\mathbb{D}) + \sup_E \frac{\mu(S(E))}{\mathrm{Cap}(E)},$$

where $\mathrm{Cap}(E)$ is the logarithmic capacity (the one for which $\mathrm{Cap}(I) \approx \log^{-1}(|I|^{-1})$ is approximatively the capacity of a small arc). In turn, $\|g\|_{\mathcal{M}(\mathcal{D})} \approx \|g\|_{H^\infty} + [\mu_g]_{CM(\mathcal{D})}$. (It is useful considering Carleson measures for $\mathcal{D}$ supported on the boundary of $\mathbb{D}$ for studying boundary values of Dirichlet functions, but we do not need them here). Following the lead of the Hardy theory, we might think that the right sobstitute of $BMOA$ in Dirichlet theory might be the space χ,

$$\|b\|_{\chi} := [\mu_b]_{CM(\mathcal{D})} = [|b'|^2 dA]_{CM(\mathcal{D})}.$$

Lacking inner/outer factorization, the analog of H^1 might be the *weak product space* $\mathcal{D} \odot \mathcal{D}$,

$$\|h\|_{\mathcal{D}\odot\mathcal{D}} = \inf\{\sum_j \|a_j\|_{\mathcal{D}} \|b_j\|_{\mathcal{D}} : \sum_j a_j b_j = h\}.$$

(For weak products in general, see [5]). Note that $H^2 \odot H^2 = H^2 \cdot H^2 = H^1$. Since $1 \in \mathcal{D}$, we have the chain of inclusions

$$H^\infty \cap \chi = \mathcal{M}(\mathcal{D}) \hookrightarrow \chi \hookrightarrow \mathcal{D} \hookrightarrow \mathcal{D} \odot \mathcal{D}.$$

Theorem 1 ([1])**.** *$(\mathcal{D} \odot \mathcal{D})^* = \chi$ under $\mathcal{D}$ pairing.*

Theorem 1 might be seen as an analog of Fefferman's Theorem in Dirichlet theory. In proving it, we found it easier passing to an equivalent formulation in terms of Hankel-type forms. Given b, holomorphic in $\mathbb{D}$, define $T_b^{\mathcal{D}}(f, g) =< fg, b >_{\mathcal{D}}$. Functional analytic considerations show that

$$\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} := \sup \frac{|T_b^{\mathcal{D}}(f, g)|}{\|f\|_{\mathcal{D}} \|g\|_{\mathcal{D}}} = \|b\|_{(\mathcal{D}\odot\mathcal{D})^*}.$$

What one has to prove is then

Theorem 2. $\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} \approx \|b\|_{\chi}$.

This is done in [1], and it might be seen as a Nehari-type theorem. It is easily seen that $\|T_b^{\mathcal{D}}\|_{\mathcal{D}\times\mathcal{D}} \lesssim \|b\|_{\chi}$. In the other direction, we use Stegenga's capacitary characterization of Carleson measures ofr $\mathcal{D}$, discrete approximation of the extremal function for the capacity of a given set and estimates of holomorphic versions of these discrete functions. A discussion of the context surrounding these theorems is in [2].

Results of similar flavor have been obtained for a few other functions spaces. Ferguson and Lacey [8] considered the Hardy space on the polydisc, while Mazya and Verbitsky [11] have, as a consequence of a more general theory, analogous results for some Sobolev spaces.

3. Related questions.

We end the abstract with some open questions.

- Is there a better, more geometric characterization of the functions belonging to χ and $\mathcal{D} \odot \mathcal{D}$?
- Are there versions of the John-Niremberg inequality [10] for functions belonging to the space χ?
- Are there analogous results for other holomorphic function spaces? The techniques used in [1] can not be easily transfered outside the Dirichlet case. It would be especially interesting to have results for the weighted Dirichlet spaces which are intermediate between Hardy and Dirichlet,
$$\|f\|^2_{\mathcal{D}_a} = \int_{\mathbb{D}} |f'(z)|^2(1-|z|^2)^a dA(z),\ 0 < a < 1,$$
as well as results for the analytic Besov spaces [14].
- We single out the above question in the special case of the Drury- the Drury-Arveson space [6] [3], in view of its importance as the analog of the Hardy space in multivariable operator theory.

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