Esercizi sui numeri complessi.

(1) Trovare le soluzioni delle seguenti equazioni in $\mathbb{C}$.

(1.1) $z^2 + 3iz - 2 = 0$; $z^2 + 3z + 2 = 0$; $z^2 + 3z + 2i = 0$

(1.2) $z^2 + z + 1 = 0$; $z^2 + z - 1 = 0$; $z^4 + z^2 - 1 = 0$

(1.3) $z^4 + 1 = 0$; $z^4 - 1 = 0$; $z^4 + i = 0$; $z^4 - i = 0$

(1.4) $z^2 - (i+1)z + i = 0$; $(z+i)^2 - (i+1)(z+i) + i = 0$

(1.5) $z^5 + 1 + i = 0$; $z^5 = 1 + i$; $z^6 + z + 3i = 0$

(1.6) $(3iz^2 + (1+i)z + 5)(z^7 + i) = 0$

(2) Trovare le soluzioni di (1.6) con parte reale positiva. Trovare quelle con parte immaginaria negativa.

(3) Sia $A = \{z \in \mathbb{C} : z^{18} = 2i + 1\}$.
Quanti sono gli elementi di A aventi parte reale negativa?

(4) Calcolare: $(1+i)^{17}$; i^{57} ; $(2i)^7$

Soluzioni e svolgimenti.

(1.1) $\boxed{z^2+3iz-2=0}$ $\Delta=(3i)^2-4\cdot(-2)=-9+8=-1=i^2$ $z=\frac{-3i\pm i}{2}=-2i,$

$\boxed{z^2+3z+2=0}$ $\Delta=9-4\cdot 2=1>0$: $z=\frac{-3\pm 1}{2}=-2,-1$

$\boxed{z^2+3z+2i=0}$ $\Delta=9-4\cdot 2i=9-4i$. Risolvo $w^2=9-4i=\sqrt{97}\cdot e^{i\theta}$ con

$\theta=\mathrm{arctg}(-4/9)=-\mathrm{arctg}(4/9)$:

$w=\pm\sqrt[4]{97}\cdot\left[\cos\left(\frac{1}{2}\mathrm{arctg}(4/9)\right)-i\sin\left(\frac{1}{2}\mathrm{arctg}(4/9)\right)\right]$ e

$z=\left\{-3\pm\sqrt[4]{97}\cdot\left[\cos\left(\frac{1}{2}\mathrm{arctg}(4/9)\right)-i\sin\left(\frac{1}{2}\mathrm{arctg}(4/9)\right)\right]\right\}\cdot\frac{1}{2}$

(1.2) $\boxed{z^2+z+1=0}$ $\Delta=1-4=-3=3i^2=(\sqrt{3}i)^2\Rightarrow z=\frac{-1\pm i\sqrt{3}}{2}$

$\boxed{z^2+z-1=0}$ $\Delta=1+4\Rightarrow z=\frac{-1\pm\sqrt{5}}{2}$

$\boxed{z^4+z^2-1=0}$ Se $w=z^2$: $w^2+w-1=0\Rightarrow w_1=\frac{-1+\sqrt{5}}{2}>0$; $w_2=\frac{-1-\sqrt{5}}{2}<0$

$z^2=\frac{-1+\sqrt{5}}{2}\Leftrightarrow z=\pm\sqrt{\frac{-1+\sqrt{5}}{2}}$; $z^2=\frac{-1-\sqrt{5}}{2}=(-1)\frac{1+\sqrt{5}}{2}=i^2\cdot\frac{1+\sqrt{5}}{2}$

$\Leftrightarrow z=\pm i\sqrt{\frac{1+\sqrt{5}}{2}}$: $z=\pm\sqrt{\frac{-1+\sqrt{5}}{2}}$ o $z=\pm i\sqrt{\frac{1+\sqrt{5}}{2}}$

(1.3) $\boxed{z^4+1=0}$; $z^4=-1=e^{i(\pi+2k\pi)}$, $k\in\mathbb{Z}$; $z=\cos\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{4}+k\frac{\pi}{2}\right)$ $k=0,1,2,3$

$\boxed{z^4-1=0}$ $z^4=1\Leftrightarrow z=1,i,-1,-i$

$\boxed{z^4+i=0}$ $z^4=-i=e^{-i\pi/2}$; $z=\cos\left(-\frac{\pi}{8}+k\frac{\pi}{2}\right)+i\sin\left(-\frac{\pi}{8}+k\frac{\pi}{2}\right)$; $k=0,1,2,3$

$\boxed{z^4-i=0}$ $z^4=i=e^{i\pi/2}$; $z=\cos\left(\frac{\pi}{8}+k\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{8}+k\frac{\pi}{8}\right)$; $k=0,1,2,3$

(1.4) $\boxed{z^2-(i+1)z+i=0}$ $\Delta=(i+1)^2-4i=i^2+2i+1-4i=-1+2i+1-4i=-2i$

$w^2=-2i=2\cdot e^{-\frac{\pi}{2}i}\Leftrightarrow w=\pm\sqrt{2}\cdot e^{-\frac{\pi}{4}i}=\pm\sqrt{2}\cdot\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right]$

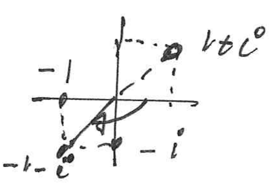
$=\pm\sqrt{2}\cdot\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=\pm(1-i)$: $z=\frac{1+i\pm(1-i)}{2}=1,i$

oss. Le soluzioni z_1,z_2 dell'equazione soddisfano $z_1+z_2=i+1$; $z_1z_2=i$

$\boxed{(z+i)^2-(i+1)(z+i)+i=0}$ Se $w=z+i$, $w^2-(i+1)w+i=0\Leftrightarrow w=1$ o $w=i$

cioè $z+i=1$ o $z+i=i$, $\boxed{z=1-i,\ z=0}$

(1.5) $\boxed{z^5+1+i=0}$ $z^5=-1-i=\sqrt{2}\cdot e^{-i\frac{3}{4}\pi}$



$z=\sqrt[10]{2}\cdot\left[\cos\left(\frac{-3}{20}\pi+\frac{2k\pi}{5}\right)+i\sin\left(-\frac{3}{20}+\frac{2k\pi}{5}\right)\right]$ $k=0,1,2,3,4$

$\boxed{z^5=1+i}=\sqrt{2}\cdot e^{i\frac{\pi}{4}}\Leftrightarrow z=\sqrt[10]{2}\cdot\left[\cos\left(\frac{\pi}{20}+\frac{2k\pi}{5}\right)+i\sin\left(\frac{\pi}{20}+\frac{2k\pi}{5}\right)\right]$; $k=0,1,2,3,4$

$\boxed{z^6+2+3i=0}$ $z^6=-2-3i=\sqrt{13}\cdot e^{i\left(\mathrm{arctg}\left(\frac{3}{2}\right)+\pi\right)}$:

$z=\sqrt[12]{13}\cdot\left[\cos\left(\frac{1}{6}\mathrm{arctg}\left(\frac{3}{2}\right)+\frac{(2k+1)\pi}{6}\right)+i\sin\left(\frac{1}{6}\mathrm{arctg}\left(\frac{3}{2}\right)+\frac{(2k+1)\pi}{6}\right)\right]$

(1•6) $\boxed{(3i\cdot z^2+(1+i)z+5)\cdot(z^7+i)=0}$ se e solo se

$3i\cdot z^2+(1+i)z+5=0$ o $z^7+i=0$

$$z^7=-i=e^{-i\frac{\pi}{2}}$$

$$z=\cos\left(-\frac{\pi}{14}+\frac{2k\pi}{7}\right)+i\sin\left(-\frac{\pi}{14}+\frac{2k}{7}\right.$$

$k=0,1,2,3,4,5,6$

moltiplico per $-i$:

$3z^2+(1-i)z-5i=0$

$\Delta=(1-i)^4+60i=-2i+60i=58i$

$=58\cdot e^{i\pi/2}$; $w^2=58\cdot e^{i\pi/2}$ $\iff$

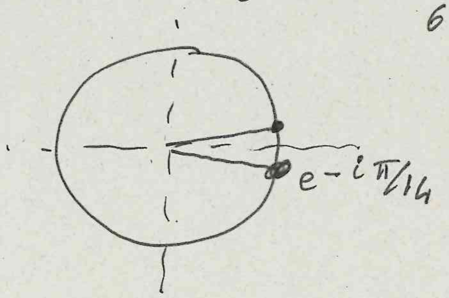
$w=\pm\sqrt{58}\cdot e^{i\pi/4}=\pm\sqrt{58}\cdot\left(\frac{1}{\sqrt2}+\frac{i}{\sqrt2}\right)$

$z=\left(-1-i\pm\sqrt{58}\cdot\left(1/\sqrt2+i/\sqrt2\right)\right)\cdot\frac{1}{6}$

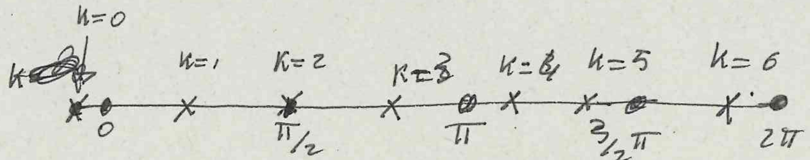
$=\left(-1-i\pm\sqrt{27}\,(1+i)\right)\cdot\frac{1}{6}$

(2) Parte reale positiva: $z=\frac{-1+\sqrt{27}}{6}+i\,\frac{-1+\sqrt{27}}{6}$

con $Re(z)=\frac{-1+\sqrt{27}}{6}$



$$-\frac{\pi}{14}+\frac{2k\pi}{7}=\frac{\pi}{14}\cdot(4k-1)$$



$\Rightarrow$

k	0	1	2	3	4	5	6
4k-1	-1	3	7	11	15	19	23

$\frac{19}{14}<\frac{3}{2}$

Hanno parte reale positiva quelle con $k=0,1,6$

Parte immaginaria negativa: $z=\frac{-1-\sqrt{27}}{6}+i\,\frac{-1-\sqrt{27}}{6}$

e $z=\cos\left(-\frac{\pi}{14}+\frac{2k\pi}{7}\right)+i\cdot\sin\left(-\frac{\pi}{14}+\frac{2k\pi}{7}\right)$ con $k=0,4,5,6$

(3) 9, ovviamente.

(4) • $(1+i)^{17}=(\sqrt2\cdot e^{i\pi/4})^{17}=\sqrt2^{17}\cdot e^{i\frac{17}{4}\pi}=\sqrt2\cdot2^8\cdot e^{i\pi/4}$

$=256\cdot\sqrt2\cdot\frac{1+i}{\sqrt2}=256\cdot(1+i)=256+256i$

• $i^{57}=i^{4\cdot14+3}=i$ [illegible] $=+i$

• $(2i)^7=2^7\cdot i^7=128\cdot i^3=-128\cdot i$.