Esercizi sugli integrali

① $\displaystyle\int_1^2 \frac{2x-7}{x^2-7x+12}\,dx$ ② $\displaystyle\int_0^{\pi/6} tg(x)\,dx$

③ $\displaystyle\int_0^{\pi} e^{2x}\cdot \sin(x)\,dx$ ④ $\displaystyle\int_{-1}^{0} \frac{dx}{\sqrt{x^2+2x+2}}$

⑤ $\displaystyle\int_{-\frac{\pi}{4}}^{0} \frac{\cos(x)}{\cos^2(x)+1}\,dx$ ⑥ $\displaystyle\int_0^1 e^{3x}\cdot \log(2+e^{3x})\,dx$

⑦ $\displaystyle\int_0^1 \frac{x^2}{6-x-x^2}\,dx$ ⑧ $\displaystyle\int_2^3 \frac{e^{2x}+3e^x}{\sqrt{e^{2x}-1}}\,dx$

Alcune domande sugli integrali.

Sia $f \in C^1(\mathbb{R},\mathbb{R})$ e sia $F:\mathbb{R}\to\mathbb{R}$, $F(x)=\int_{-1}^{x} f(t)\,dt$.
Quali delle seguenti affermazioni seguono necessariamente dalle ipotesi?

(a) Se $f \geq 0$ in $\mathbb{R}$, allora $\int_2^3 f(x)\,dx \geq 0$

(b) Se $\int_2^3 f(x)\,dx \geq 0$, allora $f \geq 0$ in $\mathbb{R}$

(c) Se $\int_2^3 f(x)\,dx \geq 0$, allora $f \geq 0$ in $[2,3]$

(d) Se $f \geq 0$ in $\mathbb{R}$, allora $F \geq 0$ in $\mathbb{R}$.

(e) Se $f \geq 0$ in $\mathbb{R}$, allora F è crescente in $\mathbb{R}$.

(f) Se F è crescente in $\mathbb{R}$, allora $f \geq 0$ in $\mathbb{R}$.

(g) Se $F \geq 0$ in $\mathbb{R}$, allora $f \geq 0$ in $\mathbb{R}$

(h) Se $\int_2^3 f(x)\,dx = 0$, allora $\forall x \in [2,3]: f(x) = 0$

(i) Se $\int^3 f(x)\,dx = 0$, allora $\exists x \in [2,3]: f(x) = 0$.

Svolgimento.

(1) Noto che $\frac{d}{dx}(x^2-7x+12) = 2x-7$, dunque,

$$\int_1^2 \frac{2x-7}{x^2-7x+12}\,dx = \int_1^2 \frac{D(x^2-7x+12)}{x^2-7x+12}\,dx = \left(\log|x^2-7x+12|\right)_1^2 =$$

$$= \log 2 - \log 6 = \log(1/3) = \boxed{-\log 3}$$

(2) $\int_0^{\pi/6} tg(x)\,dx = \int_0^{\pi/6} \frac{\sin(x)}{\cos(x)}\,dx$ Osservo che $D\cos(x) = -\sin(x)$

$$= -\int_0^{\pi/6} \frac{D\cos(x)}{\cos(x)}\,dx = -\left(\log|\cos(x)|\right)_0^{\pi/6} = -\log(\cos(\pi/6)) + \log 1$$

$$= -\log\frac{\sqrt{3}}{2} = \boxed{\log 2 - \frac{1}{2}\log 3.}$$

(3) Integro per parti due volte:

$$\int_0^{\pi} e^{2x}\cdot\sin(x)\,dx = \left(\frac{e^{2x}}{2}\cdot\sin(x)\right)_0^{\pi} - \int_0^{\pi} \frac{e^{2x}}{2}\cdot\cos(x)\,dx$$

$$= -\left[\left(\frac{e^{2x}}{2^2}\cdot\cos(x)\right)_0^{\pi} - \int_0^{\pi} \frac{e^{2x}}{2^2}\cdot(-\sin(x))\,dx\right]$$

$$= -\left(\frac{e^{2\pi}}{4}(-1) - \frac{1}{4}\right) - \frac{1}{4}\int_0^{\pi} e^{2x}\cdot\sin(x)\,dx$$

$$= \frac{e^{2\pi}+1}{4} - \frac{1}{4}\cdot\int_0^{\pi} e^{2x}\cdot\sin(x)\,dx$$

$$\Rightarrow \int_0^{\pi} e^{2x}\cdot\sin(x)\,dx\cdot\left(1+\frac{1}{4}\right) = \frac{e^{2\pi}+1}{4} \Rightarrow \int_0^{\pi} e^{2x}\cdot\sin(x)\,dx = \boxed{\frac{e^{2\pi}+1}{5}}$$

(4) Sia $P(x) = x^2+2x+2$: $\Delta = 2^2-4\cdot 2 = -4 < 0$: $P(x) = (x+1)^2+1$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{x^2+2x+2}} = \int_0^1 \frac{dx}{\sqrt{(x+1)^2+1}} \overset{A}{=}$$

Sostituisco $x+1 = \sinh(t)$

$x = -1+\sinh(t)$

$x=0 \Rightarrow t = \mathrm{ArcSinh}(1)$

$x=1 \Rightarrow t = \mathrm{ArcSinh}(2)$

$dx = \cosh(t)\,dt$

$$= \int_{\mathrm{ArcSinh}(1)}^{\mathrm{ArcSinh}(2)} \frac{\cosh(t)\,dt}{\sqrt{\sinh^2(t)+1}} = \int_{\mathrm{ArcSinh}(1)}^{\mathrm{ArcSinh}(2)} \frac{\cosh(t)}{\cosh(t)}\cdot dt$$

$$= \boxed{\mathrm{ArcSinh}(2) - \mathrm{ArcSinh}(1)}$$

(5) $$\int_{-\pi/4}^{0} \frac{\cos(x)}{\cos^2(x)+1}\,dx =$$

il numeratore ispira a considerare $\cos(x)\,dx = d\sin(x)$, voglio scrivere il denominatore in termini di $\sin x$

$$= \int_{-\pi/4}^{0} \frac{\cos(x)}{1-\sin^2(x)+1}\,dx =$$

sostituisco $y = \sin(x)$, $dy = \cos(x)\cdot dx$;
$x=0 \Rightarrow y = \sin(0) = 0$
$x = -\pi/4 \Rightarrow y = \sin(-\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$

$$= \int_{-1/\sqrt{2}}^{0} \frac{dy}{2-y^2} =$$

~~$2-y^2 = (\sqrt{2}-y)(\sqrt{2}+$~~
$2-y^2 = (\sqrt{2}-y)\cdot(\sqrt{2}+y)$

Scrivo $\frac{1}{2-y^2} \stackrel{?}{=} \frac{A}{\sqrt{2}-y} + \frac{B}{\sqrt{2}+y} = \frac{(A+B)\sqrt{2}+(A-B)y}{2-y}$

$\begin{cases} A+B = 1/\sqrt{2} \\ A-B = 0 \end{cases}$ $\begin{cases} 2A = 1/\sqrt{2} \\ B = a \end{cases}$ $\{ A = B = \frac{1}{2\sqrt{2}}$

$$= \int_{-1/\sqrt{2}} \frac{1}{2\sqrt{2}}\cdot\left(\frac{1}{\sqrt{2}-y} + \frac{1}{\sqrt{2}+y}\right) dy$$

$$= \frac{1}{2\sqrt{2}}\cdot\left(\log|\sqrt{2}+y| - \log|\sqrt{2}-y|\right)_{-1/\sqrt{2}}^{0} = \frac{1}{2\sqrt{2}}\cdot\left(\log\left|\frac{\sqrt{2}+y}{\sqrt{2}-y}\right|\right)_{-1/\sqrt{2}}^{0}$$

$$= \frac{1}{2\sqrt{2}}\cdot\left|\log(1) - \log\left(\frac{\sqrt{2}-\frac{1}{\sqrt{2}}}{\sqrt{2}+\frac{1}{\sqrt{2}}}\right)\right| = -\frac{1}{2\sqrt{2}}\cdot\log\frac{1}{5} = \boxed{\frac{1}{2\sqrt{2}}\cdot\log 5}$$

(6) $$\int_0^1 e^{3x}\cdot\log(2+e^{3x})\,dx =$$

Pongo $y = e^{3x}$; $dy = 3\cdot e^{3x}\cdot dx$,
$x=0 \Rightarrow y = 1$; $x=1 \Rightarrow y = e^3$

$$= \int_1^{e^3} \log(2+y)\,\frac{dy}{3}$$

integro per parti

$$\frac{1}{3}\left[(y+2)\log(y+2)\right]_1^{e^3} - \frac{1}{3}\int_1^{e^3}\frac{y+2}{y+2}\,dy$$

$$= \boxed{\frac{1}{3}\left[(e^3+2)\log(e^3+2) - 3\log 3\right] - \frac{1}{3}(e^3-1)}$$

(7) Sia $P(x) = x^2+x-6$: $\Delta = 1+6\times4 = 25>0$: $P(x) = (x-2)(x+3)$.

$$\int_0^1 \frac{x^2}{6-x-x^2}\,dx =$$

Divido: $x^2 = (6-x-x^2)\cdot(-1) + (6-x)$

$$= \int_0^1\left(-1+\frac{6-x}{6-x-x^2}\right)dx = -1 + \int_0^1 \frac{x-6}{x^2+x-6}\,dx =$$

$$= -1 + \int_0^1 \frac{(2x+1)\cdot 1/2 - 13/2}{x^2+x-6}\,dx = -1 + \frac{1}{2}\int_0^1 \frac{D(x^2+x-6)}{x^2+x-6}\,dx - \frac{13}{2}\int_0^1\frac{dx}{x^2+x-6}$$

$$= -1 + \frac{1}{2}\left(\log|x^2+x-6|\right)_0^1 - \frac{13}{2}\cdot\int_0^1 \frac{dx}{(x+3)(x-2)} =$$

Voglio $\dfrac{1}{(x+3)(x-2)} \overset{?}{=} \dfrac{A}{x+3} + \dfrac{B}{x-2} = \dfrac{(A+B)x+3B-2A}{(x+3)(x-2)}$,

cioè $\begin{cases} A+B=0 \\ 3B-2A=1 \end{cases}$ $\begin{cases} A=-B \\ 5B=1 \end{cases}$ $\begin{cases} A=-1/5 \\ B=1/5 \end{cases}$

$$= -1 + \frac{1}{2}(\log 4 - \log 6) - \frac{13}{2}\cdot\int_0^1 \left(\frac{1}{5}\,\frac{1}{x-2} - \frac{1}{5}\,\frac{1}{x+3}\right)dx$$

$$= -1 + \frac{1}{2}\log(4/6) - \frac{13}{10}\cdot\left(\log\left|\frac{x-2}{x+3}\right|\right)_0^1 = -1 + \frac{1}{2}\log\frac{2}{3} - \frac{13}{10}\left(\log\frac{1}{4} - \log\right.$$

$$= -1 + \frac{1}{2}(\log 2 - \log 3) - \frac{13}{10}\log\left(\frac{3}{8}\right) = -1 + \frac{1}{2}\log 2 - \frac{1}{2}\log 3 - \frac{13}{10}\log 3$$

$$+ \frac{13}{10}\cdot 3\cdot\log 2 = -1 + (\log 2)\cdot\left(\frac{1}{2} + \frac{39}{10}\right) - (\log 3)\cdot\left(\frac{1}{2} + \frac{13}{10}\right)$$

$$= \boxed{-1 + \frac{22}{5}\cdot\log 2 - \frac{9}{5}\cdot\log 3}$$

(8) $$\int_2^3 \frac{e^{2x} + 3\cdot e^x}{\sqrt{e^{2x}-1}}\,dx = \int_2^3 \frac{e^{2x}}{\sqrt{e^{2x}-1}}\,dx + 3\cdot\int_2^3 \frac{e^x}{\sqrt{e^{2x}-1}}\,dx$$

Pongo $e^{2x} = y$
$dy = 2\cdot e^{2x}\cdot dx$

Pongo $e^x = z$
$dz = e^x\,dx$

$$= \frac{1}{2}\int_{e^4}^{e^6} \frac{dy}{\sqrt{y-1}} + 3\cdot\int_{e^2}^{e^3} \frac{dz}{\sqrt{z^2-1}} =$$

Se non ricordate che $\dfrac{1}{\sqrt{z^2-1}} = D\,\mathrm{ArcCosh}(z)$ se $z>0$, sostituite $z = \cosh(t)$

$$= \frac{1}{2}\cdot\left(\frac{(y-1)^{1/2}}{1/2}\right)_{e^4}^{e^6} + 3\left(\mathrm{Arccosh}(z)\right)_{e^2}^{e^3}$$

$$= \boxed{\sqrt{e^6-1} - \sqrt{e^4-1} + 3\,\mathrm{Arccosh}(e^3) - 3\,\mathrm{Arccosh}(e^2)}$$

VERE: (a), (e), (f), (i)   FALSE: (b), (c), (d), (g), (h)