PREPARAZIONE AL II PARZIALE.

① Studiare $f(x) = |x-1| \cdot e^{-|x^2-4|}$

(dominio di funzione e derivata; derivata; limiti agli estremi; intervalli su cui f è continua; intervalli su cui f cresce/decresce) e tracciarne il grafico.

(1.1) Trovare i punti di massimo e minimo relativo di f; trovare $\sup\{f(x) : x \in \mathbb{R}\}$. f ha massimo in $\mathbb{R}$?

(1.2) Quante soluzioni ha l'equazione $f(x) = 1/2$?

② Calcolare $\lim\limits_{x \to 0^+} \dfrac{\sin(x)\left(\log(2+x^2) - \log(2)\right)}{x \sin(x^2) \cos(x)}$

~~③ Calcolare~~ $\cancel{\int_0^1 x \log\left(\frac{x^2+1}{4-x^2}\right) dx}$

④ Trovare le soluzioni di

$$(3i-2) \cdot z^5 - (2i+3) = 0$$

e indicare la loro collocazione approssimativa nel piano complesso.

③ Calcolare $\int_0^{\pi/2} \cos(t) \cdot \log\left(\dfrac{\sin^2(t)+1}{4-\sin^2(t)}\right) dt$

Svolgimento.

① $\text{Dominio}(f) = \mathbb{R}$; $f \in C(\mathbb{R})$.

$\lim_{x \to \pm\infty} f(x) = 0$, poiché $f(x) = |x| \cdot \left|1 - \frac{1}{x}\right| \cdot e^{-x^2 \left|1 - 4/x^2\right|} = \left|1 - \frac{1}{x}\right| \cdot \frac{|x|}{e^{x^2 \left|1 - 4/x^2\right|}}$ e confronto tra infiniti

osservo che

$f(1) = 0$ e $f(x) > 0 \quad \forall x \neq 1$.

Se $x \neq 1,\ x \neq \pm 2 \Rightarrow \exists\, f'(x) = \text{sgn}(x-1) \cdot e^{-|x^2-4|} + |x-1| \cdot e^{-|x^2-4|} \cdot \left(-\text{sgn}(x^2-4) \cdot 2x\right)$

$= \text{sgn}(x-1) \cdot e^{-|x^2-4|} \cdot \left(1 - \text{sgn}(x^2-4)\, 2x(x-1)\right)$

$$f'(x) = \text{sgn}(x-1) \cdot e^{-|x^2-4|} \cdot \begin{cases} (1 + 2x(x-1)) & \text{se } x^2 - 4 < 0, \text{ cioè } -2 < x < 2 \\ (1 - 2x(x-1)) & \text{se } x^2 - 4 > 0, \text{ cioè } x > 2 \text{ o } x < -2 \end{cases}$$

$\lim_{x \to 1^+} f'(x) = e^{-3} \neq -e^{-3} = \lim_{x \to 1^-} f'(x)$ DOMINIO DI f'

$\lim_{x \to 2^+} f'(x) = 1 - 4 \cdot 1 = -3 \neq 5 = \lim_{x \to 2^-} f'(x)$

$\lim_{x \to -2^+} f'(x) = -(1 \cancel{+} 4(-3)) \neq -(1 + 4(-3)) = \lim_{x \to -2^-} f'(x)$ ~~[illegible]~~

$\text{Dominio}(f') = \mathbb{R} \setminus \{1, \pm 2\}$

$$f'(x) \geq 0 \iff \begin{cases} x \in \text{Dominio}(f') = \mathbb{R} \setminus \{1, \pm 2\} \\ \underbrace{\text{sgn}(x-1)}_{A} \cdot \underbrace{e^{-|x^2-4|}}_{B} \cdot \underbrace{\left[1 - \text{sgn}(x^2-4)\, 2x(x-1)\right]}_{C} \geq 0 \end{cases}$$

$B > 0 \ \forall x$. $A > 0 \iff x > 1$ $B \geq 0 \iff \begin{cases} x^2 - 4 \geq 0 \\ 1 - 2x(x-1) \geq 0 \end{cases}$ o $\begin{cases} x^2 - 4 \geq 0 \\ 1 + 2x(x-1) \geq 0 \end{cases}$

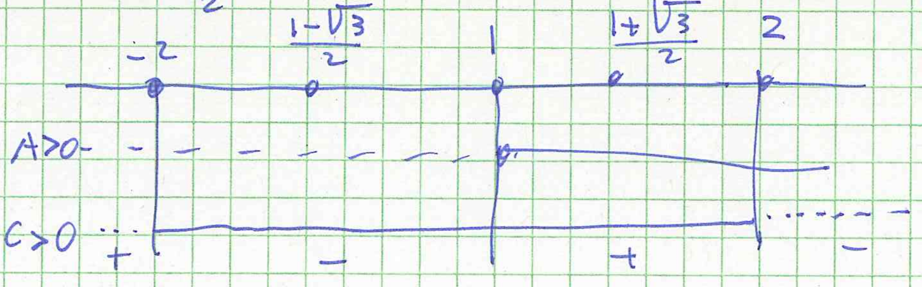
$\iff \begin{cases} x \leq -2 \text{ o } x \geq 2 \\ 0 \geq 2x^2 - 2x - 1 \end{cases}$ o $\begin{cases} -2 \leq x \leq 2 \\ 2x^2 - 2x + 1 \geq 0 \end{cases}$ $\iff \begin{cases} x \leq -2 \text{ o } x \geq 2 \\ \frac{1-\sqrt{3}}{2} \leq x \leq \frac{1+\sqrt{3}}{2} \end{cases}$ o $\left\{ -2 \leq x \leq 2 \right.$

$2y^2 - 2y - 1 = 0$

$y = \frac{1 \pm \sqrt{3}}{2}$

$\Delta < 0$

$= -1$



$f'(x) > 0 \iff x < -2$ o $1 < x < 2$

$f'(x) < 0 \iff -2 < x < 1$ o $x > 2$

f cresce in $(-\infty, -2]$ e in $[1, 2]$
decresce in $[-2, 1]$ e in $[2, +\infty)$

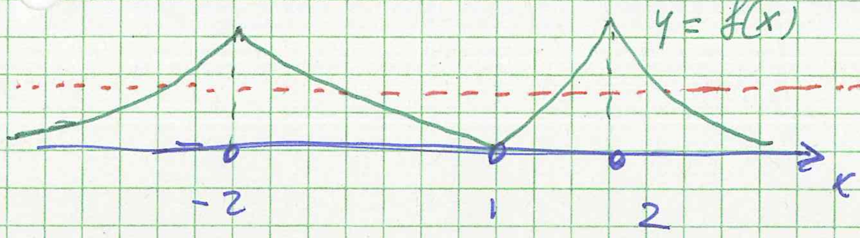


grafico di f

$x = -2$ e $x = 2$ sono p.ti di MAX. rel; $x = 1$ è p.to di min. rel.

$f(2) = 1$ e $f(-2) = 3$: $\boxed{\sup f = \max f = 3}$

$\boxed{\text{L'equazione } f(x) = 1/2 \text{ ha quattro soluzioni.}}$

(2) $\mathrm{Den}(x) = x \cdot \sin(x^2) \cos(x) \underset{x \to 0}{\sim} x \cdot x^2 \cdot 1 = x^3$ $\quad \boxed{\text{Limite } \boxed{0/0}}$

Al numeratore $\sin(x) \underset{x \to 0}{\sim} x$:

$$\frac{\sin(x) \cdot [\log(2+x^2) - \log(2)]}{x \cdot \sin(x^2) \cdot \cos(x)} \underset{x \to 0}{\sim} \frac{\log(2+x^2) - \log(2)}{x^2}$$

Sviluppo $\log(2+x^2) - \log(2) = \log\left(\frac{2+x^2}{2}\right) = \log(1+x^2)$

$$= \frac{x^2}{2} + \underset{x \to 0}{o}(x^2)$$

$$\Rightarrow \frac{\log(2+x^2) - \log(2)}{x^2} \xrightarrow[x \to 0]{} \frac{1}{2} \qquad \boxed{\lim_{x \to 0} (\ldots) = \frac{1}{2}}$$

(3) $$\int_0^1 x \cdot \log\left(\frac{x^2+1}{4-x^2}\right) dx \underset{\text{I.P.}}{=} \left[\frac{x^2}{2} \cdot \log\left(\frac{x^2+1}{4-x^2}\right)\right]_0^1 - \int_0^1 \frac{x^2}{2} \cdot \frac{d}{dx}\left(\log\left(\frac{x^2+1}{4-x^2}\right)\right) dx$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right) - \frac{1}{2}\int_0^1 x^2 \cdot \frac{d}{dx}\left[\log(x^2+1) - \log(4-x^2)\right] dx$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right) - \frac{1}{2}\int_0^1 x^2 \left(\frac{2x}{x^2+1} - \frac{-2x}{4-x^2}\right) dx$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right) - \frac{1}{2}\int_0^1 \left(\frac{x^2+1-1}{x^2+1} \cdot 2x + 2x \cdot \frac{x^2-4+4}{4-x^2}\right) dx$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right) - \frac{1}{2}\int_0^1 \left(2x - \frac{2x}{x^2+1} - 2x + \frac{8x}{4-x^2}\right) dx = \frac{1}{2}\log\left(\frac{2}{3}\right) - \left[\log(x^2+1)\right]_0^1 - 4\int_0^1 \frac{x\,dx}{4-x^2}$$

$$= \frac{1}{2}\log\left(\frac{2}{3}\right) - \log(2) + 2\left[\log(4-x^2)\right]_0^1 = \frac{1}{2}\log\left(\frac{2}{3}\right) - \log(2) + 2\log(3) - 2\log 4$$

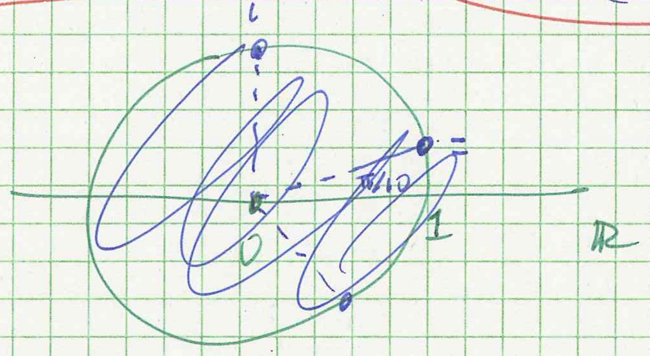
(4) $z^5 = \frac{2i+3}{3i-2} = \frac{(2i+3)(-3i-2)}{(3i-2)(-3i-2)} = \frac{6-6-9i-4i}{13} = -i$

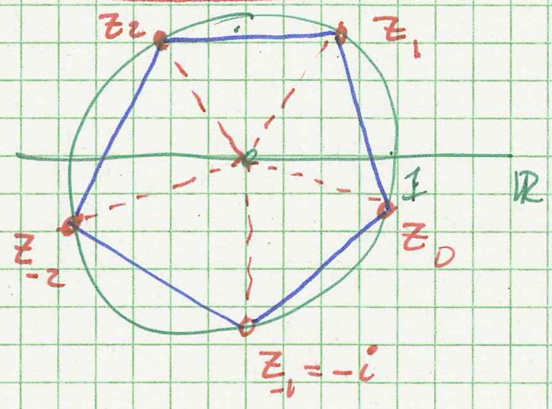
$$z^5 = -i = e^{-i\pi/2} = e^{i\left(-\frac{\pi}{2}+2k\pi\right)} \quad k=0,1,2,-1,-2$$

$\Rightarrow z = Re^{i\theta}$ con $R = \sqrt[5]{1} = 1$

e $\theta = \theta_k = -\frac{\pi}{10} + \frac{2k\pi}{5} \quad k = 0, \pm 1, \pm 2$

$$\Rightarrow \boxed{z = e^{i(-\pi/10 + 2k\pi/5)} = \cos\left(-\frac{\pi}{10}+\frac{2k\pi}{5}\right) + i\sin\left(-\frac{\pi}{10}+\frac{2k\pi}{5}\right) : k=0,\pm1,\pm2}$$





(3) $\int_0^{\pi/2} \cos(t)\cdot\log\left(\frac{1+\sin^2(t)}{4-\sin^2(t)}\right)dt = \quad x = \sin t : \; x\Big|_0^1 \text{ se } t\Big|_0^{\pi/2}$

$dx = \cos t\cdot dt$

$$= \int_0^1 \log\left(\frac{1+x^2}{4-x^2}\right)dx$$

$$= \int_0^1 \left[\log(1+x^2) - \log(2-x) - \log(2+x)\right]dx$$

I.P.

$$= \left\{x\left[\log(1+x^2) - \log(2-x) - \log(2+x)\right]\right\}_{x=0}^{1} - \int_0^1\left(x\cdot\frac{2x}{1+x^2} + \frac{x}{2-x} - \frac{x}{2+x}\right)dx$$

$$= \log 2 - \log 3 - \int_0^1\left(2 - \frac{2}{1+x^2} - 1 + \frac{2}{2-x} - 1 + \frac{2}{2+x}\right)dx$$

$$= \log 2 - \log 3 + 2\arctg(1) + 2\left(\log|x-2|\right)_0^1 - 2\left(\log|x+2|\right)_0^1$$

$$= \log 2 - \log 3 + 2\cdot\frac{\pi}{4} - 2\log 2 - 2\log 3 + 2\log 2$$

$$\boxed{= \log 2 - 3\log 3 + \frac{\pi}{2}}$$