Limiti in più variabili.

(1) Pongo $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\begin{matrix} r \geq 0 \\ \theta \in \mathbb{R} \end{matrix}$: $f(x,y) = \dfrac{r^2 \cos\theta \cdot \sin\theta}{r^{2\alpha}} =$

$$= r^{2-2\alpha} \cdot \cos\theta \cdot \sin\theta$$

Poichè ~~Quindi~~ $\exists \lim_{r\to 0} f(r\cos\theta, r\sin\theta) \iff \exists \lim_{(x,y)\to(0,0)} f(x,y)$
indipendentemente da θ

e $r^{2-2\alpha} \cdot \cos\theta \cdot \sin\theta \xrightarrow[r\to 0]{} \begin{cases} 0 & \text{se } 0 \leq \alpha < 1 \\ \cos\theta \cdot \sin\theta & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha > 1 \end{cases}$

si ha che $\exists \lim_{(x,y)\to(0,0)} f(x,y) \in \mathbb{R} \iff \boxed{0 \leq \alpha < 1}$

(2) $\begin{cases} x = \frac{r}{2}\cos\theta \\ y = \frac{r}{3}\sin\theta \end{cases}$ con $\begin{matrix} r \geq 0 \\ \theta \in \mathbb{R} \end{matrix}$: $f(x,y) = \dfrac{r^2}{r^{2\alpha}} \cdot \left(\dfrac{\cos^2\theta}{4} - \dfrac{\sin^2\theta}{9}\right)$

$$\xrightarrow[r\to 0]{} \begin{cases} 0 & \text{se } 0 \leq \alpha < 1 \\ L(\theta) \text{ dipendente da } \theta & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha > 1 \end{cases}$$

$\boxed{0 \leq \alpha < 1}$

(3) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{matrix} r \geq 0 \\ \theta \in \mathbb{R} \end{matrix}$: $f(x,y) = \dfrac{r^2}{r^\alpha} \cdot \dfrac{1}{(|\cos\theta| + |\sin\theta|)^\alpha}$

Oss. che $|\cos\theta| + |\sin\theta| \geq \delta > 0$ $(\exists \delta > 0 : \forall \theta \in \mathbb{R})$.
e che $|\cos\theta| + |\sin\theta| \leq 2 \quad \forall \theta \in \mathbb{R}$.

Quindi

$0 \leq \alpha < 2 \implies f(x,y) = r^{2-\alpha} \cdot \dfrac{1}{(|\cos\theta| + |\sin\theta|)^\alpha} \leq \dfrac{r^{2-\alpha}}{\delta^\alpha} \xrightarrow[r\to 0]{} 0$

$\alpha = 2 \implies f(x,y) = \dfrac{1}{(|\cos\theta| + |\sin\theta|)^\alpha} = L(\theta) \xrightarrow[r\to 0]{} L(\theta)$ dip. da θ

$\alpha > 2 \implies f(x,y) = \dfrac{1}{r^{\alpha-2}} \cdot \dfrac{1}{(|\cos\theta| + |\sin\theta|)^\alpha} \geq \dfrac{1}{2^\alpha} \cdot \dfrac{1}{r^{\alpha-2}} \xrightarrow[r\to 0]{} +\infty$

$\boxed{0 \leq \alpha < 2}$

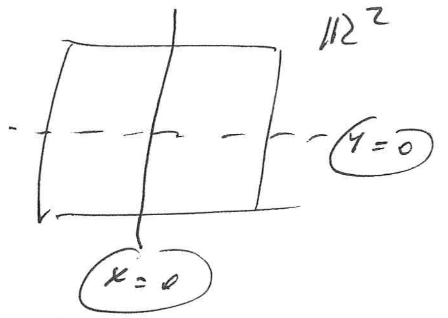
(4) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{matrix} r \geq 0 \\ \theta \in \mathbb{R} \end{matrix}$ $\Rightarrow f(x,y) = \dfrac{r^2 + r\cos\theta + r\sin\theta}{r^{2\alpha}} =$

$$= \frac{r}{r^{2\alpha}} \cdot (\cos\theta + \sin\theta + r) = \frac{r}{r^{2\alpha}}\left(\cos\theta + \sin\theta + o(1)_{r\to 0}\right)$$

$$\Rightarrow \lim_{r\to 0} f(x,y) = \begin{cases} 0 & \text{se } 0 \leq \alpha < 1/2 \\ L(\theta) = \cos\theta + \sin\theta & \text{se } \alpha = 1/2 \\ +\infty & \text{se } \alpha > 1/2 \end{cases} :$$

$\boxed{0 \leq \alpha < 1/2}$

(5) f è definita per $x \neq 0$:



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{cases} r \geq 0 \\ \theta \in \mathbb{R} \end{cases}$: $f(x,y) = \dfrac{r}{r^\alpha} \, \dfrac{\cos\theta + \sin\theta}{|\cos\theta|^\alpha}$

Sia $0 \leq \alpha < 1$: $f(x,y) = r^{1-\alpha} \cdot \dfrac{\cos\theta + \sin\theta}{|\cos\theta|^\alpha} = r^{1-\alpha} \cdot L(\theta)$

~~se $\theta = \alpha$, $L(\theta) = +\infty$; se~~

Per $\theta = 0$, $f(x,y) = r^{1-\alpha} \xrightarrow[r\to 0]{} 0$

Per $\theta = \frac{\pi}{2} - r^\beta$, $f(x,y) = r^{1-\alpha} \cdot \dfrac{\cos\left(\frac{\pi}{2} - r^\beta\right) + \sin\left(\frac{\pi}{2} - r^\beta\right)}{\left|\cos\left(\frac{\pi}{2} - r^\beta\right)\right|^\alpha}$

$$= r^{1-\alpha} \cdot \frac{\sin(r^\beta) + \cos(r^\beta)}{|\sin(r^\beta)|^\alpha} \underset{r\to 0}{\sim} \frac{r^{1-\alpha}}{r^{\alpha\beta}}$$

$= r^{1-\alpha-\alpha\beta}$ e se scelgo $\beta > 0$: $1 - \alpha - \alpha\beta < 0$ (cioè $\beta > \frac{1-\alpha}{\alpha}$), ho che

$r^{1-\alpha-\alpha\beta} \to +\infty$: il limite non esiste se $0 \leq \alpha < 1$.

Se $\alpha = 1$: $f(x,y) = \dfrac{\cos\theta + \sin\theta}{|\cos\theta|^\alpha} = L(\theta) \xrightarrow[r\to 0]{} L(\theta)$ dip. da θ.

Se $\alpha > 1$, $f(x,y) = r^{1-\alpha} \to +\infty$: $\boxed{\text{il limite non è mai in } \mathbb{R}.}$

(7) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{cases} r \geq 0 \\ \theta \in \mathbb{R} \end{cases} \Rightarrow f(x,y) = \dfrac{r^2(\cos^2\theta + 2\sin^2\theta) + o_{r\to 0}(r^2)}{r^{2\alpha}}$

$= r^{2-2\alpha} \cdot \left(1 + \sin^2\theta + o_{r\to 0}(1)\right) \xrightarrow[r\to 0]{} \begin{cases} 0 & \text{se } 0 \leq \alpha < 1 \\ L(\theta) & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha > 1 \end{cases}$

$\boxed{0 \leq \alpha < 1}$

(6) $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{cases} r \geq 0 \\ \theta \in \mathbb{R} \end{cases} \Rightarrow f(x,y) = \dfrac{r^2\cdot(\cos^2\theta + \sin^2\theta) + o(r^2)}{r^{2\alpha}}$

$= r^{2-2\alpha}\left(1 + o_{r\to 0}(1)\right) \xrightarrow[r\to 0]{} \begin{cases} 0 & \text{se } 0 \leq \alpha < 1 \\ 1 & \text{se } \alpha = 1 \\ +\infty & \text{se } \alpha > 1 \end{cases}$

$\boxed{0 \leq \alpha \leq 1}$

(8) $f(x,y) = \dfrac{\sin\left(r^2(\cos^2\theta + 2\sin^2\theta)\right)}{r^{2\alpha}}$ Taylor + fatto che $\cos^2\theta + 2\sin^2\theta$ è limitata

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ con $\begin{cases} r \geq 0 \\ \theta \in \mathbb{R} \end{cases}$ $= \dfrac{r^2(\cos^2\theta + 2\sin^2\theta) + o_{r\to 0}(r^2)}{r^{2\alpha}}$

$= r^{2-2\alpha}\cdot\left(1 + \sin^2\theta + o_{r\to 0}(1)\right)$ come in (7):

$\boxed{0 \leq \alpha < 1}$

(9) Ovviamente $\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0 \quad \forall \alpha \geq 0$

(10) $f(x,y) = \dfrac{4x^2 + 4xy + 3y^2}{(x^2+y^2)^\alpha} = r^{2-2\alpha}\left(4\cos^2\theta + 3\cos\theta\cdot\sin\theta + 3\sin^2\theta\right)$

coordinate polari come sopra

Se $0 \leq \alpha < 1$, $\exists \lim_{(x,y)\to(0,0)} f(x,y) = 0$; se $\alpha = 1$; $\lim f(x,y) = L(\theta)$ che dipende da θ

Se $\alpha > 1$, per $\theta = 0$, $\lim f(r,0) = \lim_{r\to 0} 4\cdot r^{2-2\alpha} = +\infty$: $\boxed{0 \leq \alpha < 1}$

(11) $f(x,y) = (1-\alpha) \cdot r^{1-2\alpha} \cdot (\cos\theta + \sin\theta)$:

in coordinate polari

$\exists \lim_{r \to 0} f(x,y) \in \mathbb{R}$ se $\alpha = 1$

o se $1 - 2\alpha > 0$:

$$\boxed{0 \leq \alpha < 1/2 \quad \text{o} \quad \alpha = 1}$$