Prova scritta complessiva di Analisi Matematica II
Ingegneria Edile-Architettura, 2010/11

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(1) **[14 pti]** Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \{(x, y, z) : \ x^2 + y^2 \geq 1, \ 4 \leq x^2 + y^2 + z^2 \leq 9, \ z \geq 0\}$. **(1.1)**Fare un disegno *qualitativo* di Ω (usando coordinate cilindriche viene forse meglio).

(1.2) Parametrizzare $\partial\Omega$ e dire se la parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$.

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x, y, z) = (y, -x, z)$.

(1.5) . Sia $\Sigma = \{(x, y, z) : \ x^2 + y^2 + z^2 = 4, \ x^2 + y^2 \geq 1, \ z \geq 0\}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la scelta μ della normale a Σ per cui $\mu = -\nu$ (ν essendo la normale di cui al punto (1.2)).

(1.6) Calcolare la circuitazione $\iint_\Sigma (\nabla \times G) \cdot \mu d\sigma$, con $G \in C^1(\Sigma, \mathbb{R}^3)$, $G(x, y, z) = (x, y, z)$.

(2) [3 pti] Dire per quali valori del parametro $\alpha \in \mathbb{R}$ il campo $F : \mathbb{R}^2 \to \mathbb{R}^2$ è esatto e, per quei valori, calcolarne il potenziale; dove

$$F(x, y) = \cancel{(e^x[y\cos(y) + x\sin(y), x\cos(y) + \alpha y\sin(y)])}.$$

$$= \left(e^x \cdot (y\cos y + x\sin y);\ e^x \cdot (x\cos(y) + \alpha \cdot y\sin(y))\right)$$

(3) [4 pti] Sia $A = \{(x, y) : x^2 + y^2 \leq 1,\ x \leq 0 \leq y\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A e^{-x^2-y^2}\, dxdy.$$

(4) [6 pti] Classificare i punti critici di $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = (x-1)(x+1)(x^2 + y^2 - 4)$.

(5) [3 pti] Trovare l'integrale generale di

$$\dot{x} - tx = t$$

(6) [3 pti] Siano $f \in C^1(\mathbb{R}, \mathbb{R})$ e sia $G : \mathbb{R}^3 \to \mathbb{R}^3$ definita da

$$G(u, v, t) = (f(t)\cos(u), f(t)\sin(u), v).$$

Sia ora $h : \mathbb{R}^3 \to \mathbb{R}$, $h \in C^1$, $h = h(x, y, z)$, e si definisca

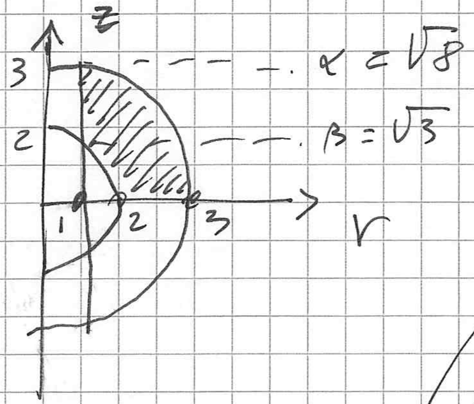
$$\varphi(u, v, t) = h(G(u, v, t)).$$

Calcolare $\frac{\partial \varphi}{\partial t}(u, v, t)$ e $\frac{\partial \varphi}{\partial t}(\pi/4, e, 0)$.

(1a) $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \end{cases}$ $r \geq 0$, $|\theta| \leq \pi$ $r^2 \geq 1$

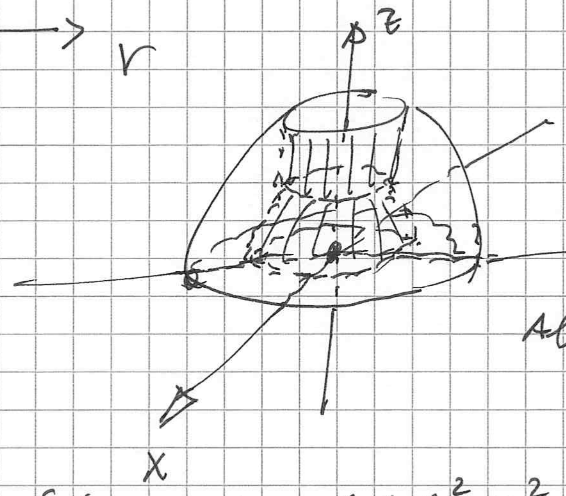
$4 \leq r^2 + z^2 \leq 9$

$z \geq 0$



$(x, y, z) \in \Omega \iff r \geq 1;\ z \geq 0$

e $4 \leq r^2 + z^2 \leq 9$



Trovo α: $\begin{cases} r = 1 \\ r^2 + z^2 = 9 \\ z \geq 0 \end{cases}$

$\{z^2 = 8,\ z \geq 0$

$\alpha = \sqrt{8}$

Allo stesso modo, $\beta = \sqrt{3}$

(1b) $\Sigma_1 = \{(x, y, 0) : 4 \leq x^2 + y^2 \leq 9\}$ $A_1 \xrightarrow{\Phi_1} \Sigma_1$ ✓

$\Phi_1(x, y) = (x, y, 0)$; $A_1 = \{(x, y) : 4 \leq x^2 + y^2 \leq 9\} \subseteq \mathbb{R}^2$

$\Sigma_2 = \{(x, y, z) : x^2 + y^2 = 1;\ \sqrt{3} \leq z \leq \sqrt{8}\}$; $A_2 \xrightarrow{\Phi_2} \Sigma_2$

$\Phi_2(\theta, z) = (\cos\theta, \sin\theta, z)$; $A_2 = \{(\theta, z) : |\theta| \leq \pi;\ \sqrt{3} \leq z \leq \sqrt{8}\}$.

$\Sigma_3 = \{(x, y, z) : 4 = x^2 + y^2 + z^2;\ z \geq 0,\ x^2 + y^2 \geq 1\}$; $A_3 \xrightarrow{\Phi_3} \Sigma_3$

$\Phi_3(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{4 - r^2})$; $A_3 = \{(r, \theta) : |\theta| \leq \pi,\ 1 \leq r \leq 2\}$

$\Sigma_4 = \{(x, y, z) : 9 = x^2 + y^2 + z^2;\ z \geq 0;\ x^2 + y^2 \geq 1\}$; $A_4 \xrightarrow{\Phi_4} \Sigma_4$

$\Phi_4(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{9 - r^2})$; $A_4 = \{(r, \theta) : |\theta| \leq \pi,\ 1 \leq r \leq 3\}$

$\partial_x \Phi_1 \times \partial_y \Phi_1(x, y) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: non comp.

$\partial_\theta \Phi_2 \times \partial_z \Phi_2(\theta, z) = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \overset{\text{se } \theta = 0}{=} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$: non comp.

$(\partial_r \Phi_3 \times \partial_\theta \Phi_3)(r, \theta) = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \frac{-r}{\sqrt{4 - r^2}} \\ -r\sin\theta & -r\cos\theta & 0 \end{vmatrix} \overset{\text{se } \theta = 0}{=} \begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{r}{\sqrt{4 - r^2}} \\ 0 & 1 & 0 \end{vmatrix} = \left(\frac{r}{\sqrt{4 - r^2}};\ 0;\ 1\right)$: non comp.

$\left(\partial_r \Phi_u \times \partial_\theta \Phi_u\right)(r,\theta) \underset{\text{se } \theta=0}{=} \left(\frac{r}{\sqrt{9-r^2}};\ 0;\ 1\right)$ è compo.

(1.3) Uso il Teorema della divergenza:

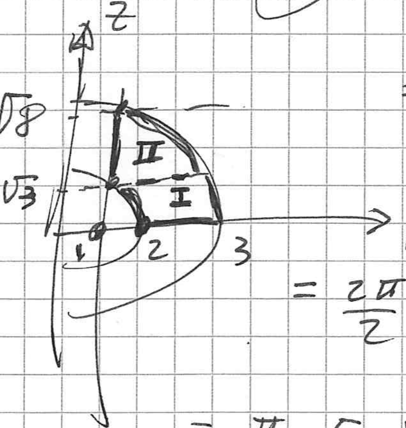
$$\iint_{\partial\Omega} F\cdot\nu\, d\sigma = \iiint_{\Omega} \operatorname{div} F(x,y,z)\, dx\, dy\, dz$$

(1.4) $F(x,y,z) = (y,-x,z) \Rightarrow \operatorname{div} F(x,y,z) = 1$

$$\Rightarrow \iint_{\partial\Omega} F\cdot\nu\, d\sigma = \iiint_{\Omega} dx\,dy\,dz = \int_{-\pi}^{\pi} d\theta \int_{\{(r,z):\ r\geq 1;\ z\geq 0;\ 4\leq r^2+z^2\leq 9\}} r\,dr\,dz \cdot 1$$

cambio in variabili cilindriche; quelle di (1.1)

$$= \underbrace{2\pi \cdot \int_0^{\sqrt3} dz \int_{\sqrt{4-z^2}}^{\sqrt{9-z^2}} r\,dr}_{\text{I}} + \underbrace{2\pi \cdot \int_{\sqrt3}^{\sqrt8} dz \int_1^{\sqrt{9-z^2}} r\,dr}_{\text{II}}$$



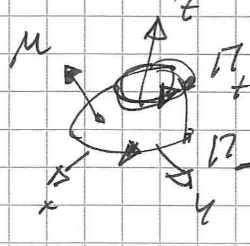
$$= 2\pi\cdot\int_0^{\sqrt3}\left(\frac{r^2}{2}\right)_{\sqrt{4-z^2}}^{\sqrt{9-z^2}} dz + 2\pi\int_{\sqrt3}^{\sqrt8}\left(\frac{r^2}{2}\right)_1^{\sqrt{9-z^2}} dz$$

$$= \frac{2\pi}{2}\cdot\int_0^{\sqrt3}(9-z^2)-(4-z^2)\,dz + \frac{2\pi}{2}\int_{\sqrt3}^{\sqrt8}(9-z^2)-1\,dz$$

$$= \pi\cdot 5\cdot\sqrt3 + \pi\cdot(\sqrt8-\sqrt3)\,8 - \pi\left(\frac{z^3}{3}\right)_{\sqrt3}^{\sqrt8}$$

$$= \pi\, 8\sqrt8 - \pi\, 3\sqrt3 - \frac{\pi}{3}\left(\sqrt8^3 - \sqrt3^3\right)$$

$$= \frac{2}{3}\pi\, 8\sqrt8 - \frac{2}{3}\pi\, 3\sqrt3 .$$



(1.5) $\Sigma = \Sigma_3$ di (1.2) $\partial\Sigma = \Gamma_+ \cup \Gamma_-$

$\Gamma_+ = \{(x,y,z):\ x^2+y^2=1;\ x^2+y^2+z^2=4;\ z\geq 0\}$

$= \{(x,y,z):\ x^2+y^2=1;\ z=\sqrt3\}$; $\gamma_+(\theta) = (\cos\theta, \sin\theta, \sqrt3)$

$\gamma_+ : [-\pi,\pi] \to \Gamma_+$ parametrizza Γ_+ come in figura; γ_+ non è compo. con μ.

$\Gamma_- = \{(x,y,z) : x^2+y^2 = 4 ; z = 0\}$; $\gamma_-(\theta) = (2\cos\theta, 2\sin\theta, 0)$

$\gamma_- : [-\pi, \pi] \to \Gamma_-$ e γ_- è compatibile con μ.

1.6 $\nabla \times G = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$ se $G = (P, Q, R)$

$$= \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \Rightarrow \iint_\Sigma (\nabla \times G) \cdot \mu \, d\sigma = 0$$

(2) $\partial_y P(x,y) = \partial_y [e^x \cdot (y\cos y + x \sin y)]$

$= e^x \cdot (\cos y - y \sin y + x \cos y)$ e

$\partial_x Q(x,y) = \partial_x [e^x \cdot (x\cos y + \alpha y \sin y)]$

$= e^x (x \cos y + \alpha y \sin y) + e^x \cos y$

quindi $\partial_y P = \partial_x Q \iff \alpha = -1$, nel qual caso

F è chiuso su $\mathbb{R}^2$, quindi è esatto.

Se $\varphi : \mathbb{R}^2 \to \mathbb{R}$ è un potenziale, $\nabla\varphi = (P, Q)$, allora

$$\varphi(x,y) = \int P(x,y)\,dx =$$

$= \int e^x (y \cos y + x \sin y)\,dx = y \cdot \cos y \int e^x\,dx + \sin y \int e^x x\,dx$

$= y \cdot \cos(y)\, e^x + \sin y \left(e^x \cdot x - \int e^x\,dx\right) =$

$= y \cdot \cos(y)\, e^x + e^x \cdot x \cdot \sin(y) - e^x \sin(y) + A(y)$

e devo avere che $e^x \cdot (x \cos(y) - y \sin(y)) = Q(x,y) \overset{*}{=}$

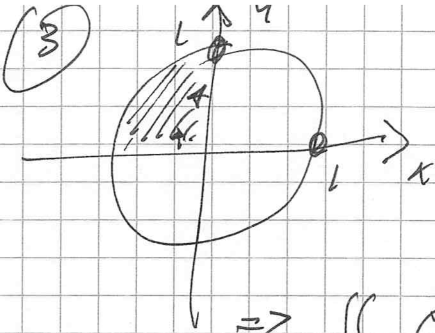
$= \partial_y \varphi(x,y) = \cos y \cdot e^x - y \sin y\, e^x + e^x \cdot x \cos y - e^x \cos y$

$+ A'(y) = e^x (x \cdot \cos(y) - y \sin y) + A'(y)$:

$Q = \partial_y \varphi \iff A'(y) = 0$, ho quindi potenziali

$\varphi(x,y) = y \cos(y)\, e^x + e^x \cdot x \cdot \sin(y) - e^x \sin(y) + c$, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, con $c \in \mathbb{R}$.

(3)



Pongo $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad r \geq 0,\ 0 < \theta \leq \pi$

$(x,y) \in A \iff 0 \leq r \leq 1$ e $\frac{\pi}{2} \leq \theta \leq \pi$

$$\Rightarrow \iint_A e^{-(x^2+y^2)}\,dx\,dy = \int_{\pi/2}^{\pi} d\theta \cdot \int_0^1 r\,dr \cdot e^{-r^2}$$

$$= \frac{\pi}{2} \cdot \left(\frac{-e^{-r^2}}{2} \right)_0^1 = \frac{\pi}{4} \cdot \left(1 - e^{-1}\right)$$

(4) $f_x = (x+1)(x^2+y^2-4) + (x-1)(x^2+y^2-4) + (x-1)(x+1)\,2x$

$f_y = 2y \cdot (x-1)(x+1)$

$$\nabla f(x,y) = 0 \iff \begin{cases} y = 0 \\ 0 = (x+1)(x^2-4) + (x-1)(x^2-4) + (x-1)(x+1) \cdot 2x \end{cases}$$

$$= 2x(x^2-4) + (x^2-1)\cdot 2x = 2x\cdot(2x^2-5)$$

o se $\begin{cases} x-1 = 0 \\ 2\cdot(y^2-3) = 0 \end{cases}$ o se $\begin{cases} x+1 = 0 \\ -2(y^2-3) = 0 \end{cases}$

P.ti critici: $(0,0)$; $\left(\pm\sqrt{5/2}, 0\right)$; $(1, \pm\sqrt{3})$; $(-1, \pm\sqrt{3})$

$f_{xx} = 2(x^2+y^2-4) + 2x\cdot[(x+1) + (x-1) + (x+1)(x-1)]$

$f_{xy} = 2y \cdot 2x \qquad f_{yy} = 2(x-1)(x+1)$

$\text{Hess} f(\pm 1, \pm\sqrt{3}) = \begin{pmatrix} 0 & A \\ A & B \end{pmatrix}$ con $A = \pm 4\sqrt{3} \neq 0$, det. negativo.

$\pm(1, \sqrt{3})$ e $\pm(-1, \sqrt{3})$ sono punti di sella.

$\text{Hess} f(0,0) = \begin{pmatrix} -8 & 0 \\ 0 & -2 \end{pmatrix}$ def. neg. p.to max. loc.

$\text{Hess}\left(\pm\sqrt{5/2}; 0\right) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ con $B > 0$ e $A > 0$: p.ti min. loc.

(5) E' un'equazione lineare ~~[illegible]~~ del I ordine.

$$a(t) = -t \Rightarrow A(t) = \int -t\,dt = -\frac{t^2}{2}.$$

EQUAZ.

$$\underbrace{\dot{x}(t) \cdot e^{-t^2/2} + (-t) \cdot e^{-t^2/2} \cdot x(t)}_{\left[x(t) \cdot e^{-t^2/2}\right]^{\cdot}} = t \cdot e^{-t^2/2}$$

$$\Leftrightarrow \quad x(t) \cdot e^{-t^2/2} = \int t \cdot e^{-t^2/2}\,dt = -e^{-t^2/2} + K$$

$$\Leftrightarrow \quad x(t) = K \cdot e^{t^2/2} - 1, \text{ con } K \in \mathbb{R},$$

$$x: \mathbb{R} \to \mathbb{R}$$

(6) $$\partial_t \varphi(u, v, t) = \partial_x h(G(u, v, t)) \cdot f'(t) \cos(u)$$

$$+ \partial_y h(G(u, v, t)) \cdot f'(t) \cdot \sin(u)$$

$$G(\pi/4; e; 0) = \left(f(0) \frac{1}{\sqrt{2}}, f(0) \frac{1}{\sqrt{2}}; e\right)$$

$$\Rightarrow \partial_t \varphi(\pi/4; e; 0) = \partial_x h\left(\frac{f(0)}{\sqrt{2}}, \frac{f(0)}{\sqrt{2}}, e\right) \cdot \frac{f'(0)}{\sqrt{2}}.$$

$$+ \partial_y h\left(\frac{f(0)}{\sqrt{2}}, \frac{f(0)}{\sqrt{2}}, e\right) \cdot \frac{f'(0)}{\sqrt{2}}.$$