Prova scritta di Analisi Matematica II (9/1/2012)

Nome..**Cognome**.. **Matricola**................

Prova orale verso: *l'inizio* | *la fine* dell'appello; **non** nel *mattino* | *pomeriggio* del giorno (Cancellare la voce che non interessa).

(1) [**14 pti**] Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \left\{(x,y,z):\ -1 \leq z \leq 7 - \sqrt{\frac{x^2}{25}+\frac{y^2}{9}},\ \frac{x^2}{25}+\frac{y^2}{9} \leq 1\right\}$.

(1.1) Fare un disegno *qualitativo* di Ω.

(1.2) Parametrizzare $\partial\Omega$ e dire se la parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$. (Nella formula devono apparire, magari iterati, solo integrali di una variabile).

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x,y,z) = \left(\frac{x}{5}e^z, \frac{y}{3}e^z, 0\right)$.

(1.5) . Sia $\Sigma = \left\{(x,y,z) \in \partial\Omega :\ \frac{x^2}{25}+\frac{y^2}{9} = 1\right\}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la scelta μ della normale a Σ per cui $\mu = -\nu$ (ν essendo la normale di cui al punto (1.2)).

(1.b) Calcolare $\iint_\Sigma (\nabla \times G) \cdot \nu d\sigma$, con $G \in C^1(\Sigma, \mathbb{R}^3)$, $G(x, y, z) = (-y(z+1), x(z+1), 0)$.

(2) **[3 pti]** Dire per quali valori del parametro $\alpha \in \mathbb{R}$ il campo $F : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ è esatto e, per quei valori, calcolarne il potenziale; dove

$$F(x,y) = \left(\frac{\alpha x}{(x^2+y^2)^9}, \frac{y}{(x^2+y^2)^9} \right).$$

(3) **[4 pti]** Sia $A = \left\{(x,y) : |5x+3y| \leq \frac{\pi}{2},\ |3x-5y| \leq \frac{\pi}{2}|\right\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A \cos(5x+3y)dxdy.$$

$$y'' + 9y = \sin(3x) + e^{3x}$$

(5) [3 pti] Sia $\varphi \in C^1(\mathbb{R}^3, \mathbb{R})$ e sia $F \in C^1(\mathbb{R}^3, \mathbb{R}^3)$. Sia $\times$ il prodotto vettoriale in $\mathbb{R}^3$. Calcolare

$$\nabla \times (\varphi F) - (\nabla \varphi) \times F.$$

Che relazione ha l'espressione ottenuta con $\nabla \times F$?

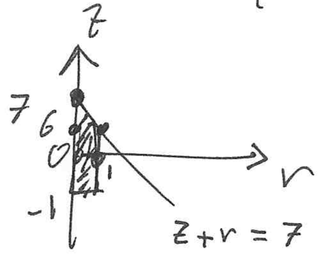
(6) [3 pti] Classificare i punti critici di $f(x, y) = (x^2 - 25)(9x^2 - 25y^2) + 15$.

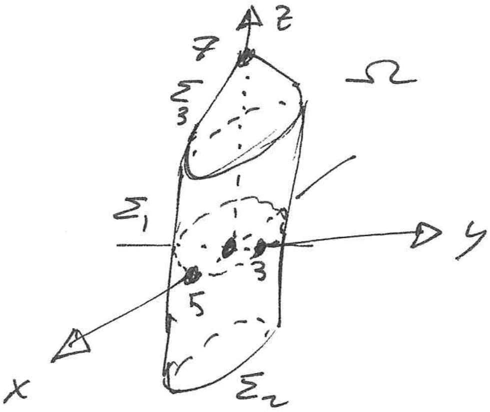
AM2 - Scritto del 9/1/2012. 1

(1) (1.1) $\Omega = \left\{(x,y,z) : -1 \le z \le 7 - \sqrt{\frac{x^2}{25}+\frac{y^2}{9}} \ ; \ \frac{x^2}{25}+\frac{y^2}{9} \le 1\right\}.$

Pongo $\begin{cases} \frac{x}{5} = r\cos\theta \\ \frac{y}{3} = r\sin\theta \end{cases}$ con $r \ge 0; \ \theta \in [-\pi, \pi]$

$(x,y,z) \in \Omega \iff \left|\begin{array}{l} r \ge 0; \ \theta \in [-\pi, \pi] \\ -1 \le z \le 7 - r; \ r^2 \le 1 \end{array}\right.$





(1.2) $A_1 = \{(z,\theta) : -1 \le z \le 6; -\pi \le \theta \le \pi\} \xrightarrow{\Phi_1} \mathbb{R}^3; \ \Phi_1(A_1) = \Sigma_1 = \left\{(x,y,z) : \frac{x^2}{25}+\frac{y^2}{9} = 1; \ -1 \le z \le 7\right\}$

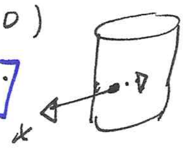
$\Phi_1(z,\theta) = (5\cos\theta, 3\sin\theta, z)$

$(\partial_z \Phi_1 \times \partial_\theta \Phi_1)(z,\theta) = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -5\sin\theta & 3\cos\theta & 0 \end{vmatrix} = (-3\cos\theta, -5\sin\theta, 0)$

se $\theta = 0$

$(-3, 0, 0)$

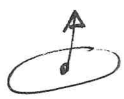
$\boxed{\text{non comp.}}$



$A_2 = \{(r,\theta) : 0 \le r \le 1; |\theta| \le \pi\} \xrightarrow{\Phi_2} \mathbb{R}^3; \ \Phi_2(A_2) = \Sigma_2 = \left\{(x,y,-1) : \frac{x^2}{25}+\frac{y^2}{9} \le 1\right\}$

$\Phi_2(r,\theta) = (5r\cos\theta, 3r\sin\theta, -1)$

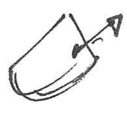
$(\partial_r \Phi_2 \times \partial_\theta \Phi_2)(r,\theta) = \begin{vmatrix} i & j & k \\ 5\cos\theta & 3\sin\theta & 0 \\ -5r\sin\theta & 3r\cos\theta & 0 \end{vmatrix} = (0, 0, 15r)$ $\boxed{\text{non comp.}}$



$A_3 = \{(r,\theta) : 0 \le r \le 1; |\theta| \le \pi\} \xrightarrow{\Phi_3} \mathbb{R}^3; \ \Phi_3(A_3) = \Sigma_3 = \left\{(x,y,z) : 7 - z = \sqrt{\frac{x^2}{25}+\frac{y^2}{9}} \le 1\right\}$

$\Phi_3(r,\theta) = (5r\cos\theta, 3r\sin\theta, 7-r)$

$(\partial_r \Phi_3 \times \partial_\theta \Phi_3)(r,\theta) = \begin{vmatrix} i & j & k \\ 5\cos\theta & 3\sin\theta & -1 \\ -5r\sin\theta & 3r\cos\theta & 0 \end{vmatrix} = (3r\cos\theta, 5r\sin\theta, 15r)$ $\boxed{\text{compatibili}}$



(1.3) Posso utilizzare (1.2): $\iint_{\partial\Omega} F\cdot\nu \, d\sigma =$

$$\boxed{\begin{aligned} &= -\int_{-1}^{6} dz \int_{-\pi}^{\pi} d\theta \, F(5\cos\theta, 3\sin\theta, z)\cdot(-3\cos\theta, -5\sin\theta, 0) \\ &\quad - \int_{0}^{1} dr \int_{-\pi}^{\pi} d\theta \, F(5r\cos\theta, 3r\sin\theta, -1)\cdot(0,0,15r) \\ &\quad + \int_{0}^{1} dr \int_{-\pi}^{\pi} d\theta \, F(5r\cos\theta, 3r\sin\theta, 7-r)\cdot(3r\cos\theta, 5r\sin\theta, 15r) \end{aligned}}$$

Posso utilizzare il Teorema della Divergenza (non è necessario, ma serve per (1.4))

$$= \iiint_{\Omega} \mathrm{div}F(x,y,z)\, dx\, dy\, dz$$

Introduco coordinate cilindriche $z = z$, $x = 5r\cos\theta$, $y = 3r\sin\theta$; $dx\,dy\,dz = 15\, r\, dr\, d\theta\, dz$

$$= \iiint \mathrm{div}F(5r\cos\theta, 3r\sin\theta, z)\cdot 15r\, dr\, d\theta\, dz$$

$\{(r,\theta,z): |\theta|\le\pi;\ 0\le r\le 1;\ -1\le z\le 7-r\}$

$$\boxed{= \int_{-\pi}^{\pi} d\theta \cdot \int_{0}^{1} dr \cdot \int_{-1}^{7-r} dz \cdot \mathrm{div}F(5r\cos\theta, 3r\sin\theta, z)\cdot 15r.}$$

(1.4) $F(x,y,z) = \left(\frac{x}{5}e^z, \frac{y}{3}e^z, 0\right) \Rightarrow \mathrm{div}F(x,y,z) = \frac{1}{5}e^z + \frac{1}{3}e^z$

$$= \frac{8}{15}e^z \Rightarrow \iint_{\partial\Omega} F\cdot\nu\, d\sigma =$$

$$= \int_{-\pi}^{\pi} d\theta \cdot \int_0^1 dr \cdot \int_{-1}^{7-r} dz \; \frac{15\cdot r\cdot 8\cdot e^z}{15} = 2\pi\cdot 8\cdot \int_0^1 r\cdot\left(e^{7-r} - e^{-1}\right) dr$$

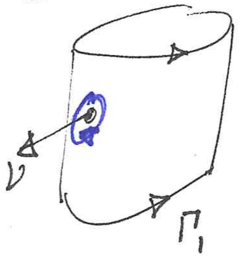
$$= 16\pi\cdot\left[e^7\left\{\left(-re^{-r}\right)\Big|_0^1 - \int_0^1 \left(-e^{-r}\right)dr\right\} - e^{-1}\left(\frac{r^2}{2}\right)\Big|_0^1\right]$$

$$= 16\pi\cdot\left[e^7\left\{\left(-e^{-1}\right) + 1 - e^{-1}\right\} - \frac{e^{-1}}{2}\right] =$$

$$= \boxed{16\pi\cdot\left[e^7\left(1-2e^{-1}\right) - \frac{e^{-1}}{2}\right].}$$

(1.5) $\Sigma = \Sigma_1$ del punto (1.2). $\Gamma_1(\theta) = (5\cos\theta, 3\cos\theta, -1)$

$\Gamma_2(\theta) = (5\cos\theta, 3\cos\theta, 6)$



$[-\pi,\pi] \xrightarrow{\Gamma_1, \Gamma_2} \mathbb{R}^3$; $\dot\Gamma_1(\theta) = \dot\Gamma_2(\theta) = (-5\sin\theta, 3\cos\theta, 0)$

Γ_1 è comp. con ν $\Rightarrow$ $\boxed{\Gamma_1 \text{ non è comp. con } \mu}$

$\boxed{\Gamma_2 \text{ è comp. con } \mu}$

(iii) Per lo Stokes ho che $\iint_{\Sigma} (\nabla \times G)\cdot \mu \, d\sigma = \int_{(\partial\Sigma, \mu)} G(s)\cdot ds$

$$= -\int_{\Gamma_1} G(s)\cdot ds + \int_{\Gamma_2} G(s)\cdot ds =$$

$$= -\int_{-\pi}^{\pi} (-3\sin\theta((-1)+1), 5\cos\theta((-1)+1), 0)\cdot(-5\sin\theta, 3\cos\theta, 0)\, d\theta$$

$$+ \int_{-\pi}^{\pi} (-3\sin\theta(6+1), 5\cos\theta(6+1), 0)\cdot(5\sin\theta, 3\cos\theta, 0)\, d\theta$$

$$= \int_{-\pi}^{\pi} (15\cdot 7\cdot \sin^2\theta + 15\cdot 7\cdot \cos^2\theta)\, d\theta = 15\cdot 7\cdot 2\pi = \boxed{210\cdot\pi}.$$

(2) Per $\alpha = 1$ il campo è radiale, quindi esatto.
In generale, F è chiuso $\iff 0 = \partial_y\left(\frac{\alpha x}{(x^2+y^2)^9}\right) - \partial_x\left(\frac{y}{(x^2+y^2)^9}\right)$

$$= \frac{\alpha x(-9)\cdot 2y}{(x^2+y^2)^{10}} - \frac{y\cdot(-9)\cdot 2x}{(x^2+y^2)^{10}} \iff \alpha = 1.$$

$F(x,y) = \nabla\varphi(x,y)$ con $\varphi(x,y) = f(x^2+y^2)$

e con $f: (0,+\infty) \to \mathbb{R}$.

$$\Rightarrow \frac{x}{(x^2+y^2)^9} = \partial_x\left(f(x^2+y^2)\right) = f'(x^2+y^2)\cdot 2x, \text{ cioè}$$

$$f'(x^2+y^2) = \frac{1}{2\cdot(x^2+y^2)^9}; \text{ ovvero } f'(t) = \frac{1}{2t^9}$$

$$\Rightarrow f(t) = -\frac{1}{16\cdot t^8} + \text{costante}.$$

$\boxed{\varphi(x,y) = -\frac{1}{16}\cdot\frac{1}{(x^2+y^2)^8} + \text{costante}}$ dà tutti i potenziali di F.

3) $\iint_A \cos(5x+3y)\,dx\,dy =$

Pongo $\begin{cases} 5x+3y = u \\ 3x-5y = v \end{cases}$

$\begin{pmatrix} u \\ v \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix}$ con

$JF(x,y) = \begin{pmatrix} 5 & 3 \\ 3 & -5 \end{pmatrix}$

$\Rightarrow |\det(JF(x,y))| = 25+9 = 34$

$\Rightarrow du\,dv = 16 \cdot dx\,dy$

$= \iint \cos(u) \dfrac{du\,dv}{34} =$

$\{(u,v) : |u| \le \frac{\pi}{2} ; |v| \le \pi/2\}$

$= \int_{-\pi/2}^{\pi/2} dv \cdot \int_{-\pi/2}^{\pi/2} \cos(u)\,du / 34$

$= \dfrac{\pi}{34} \cdot [\sin(u)]_{-\pi/2}^{\pi/2} = \dfrac{2\pi}{34} = \dfrac{\pi}{17}$

4) $y'' + 9y = \sin(3x) + e^{3x}$

Omogenea: $z'' + 9z = 0$ Eq. caratt. $\lambda^2 + 9 = 0$ $\lambda = \pm 3i$

$z(x) = A\cdot\cos(3x) + B\sin(3x)$: int. gen. omog.: risonanza con $\sin(3x)$.

Provo con $y(x) = x\cdot[C\cdot\cos(3x) + D\cdot\sin(3x)] + k\cdot e^{3x}$

$y'(x) = C\cdot\cos(3x) + D\cdot\sin(3x) + 3x\cdot[-C\cdot\sin(3x) + D\cdot\cos(3x)] + 3k e^{3x}$

$y''(x) = 6\cdot[-C\cdot\sin(3x) + D\cdot\cos(3x)] - 9x\cdot[C\cdot\cos(3x) + D\cdot\sin(3x)] + 9\cdot k\cdot e^{3x}$

Sostituisco:

$y'' + 9y = 6\cdot[-C\cdot\sin(3x) + D\cdot\cos(3x)] + 18\cdot k\cdot e^{3x}$: $k = \frac{1}{18}$, $D = 0$, $C = -\frac{1}{6}$

$$\boxed{y(x) = A\cdot\cos(3x) + B\cdot\sin(3x) + \frac{1}{18}e^{3x} - \frac{1}{6}\cdot x\cdot\cos(3x)}$$

è l'int. gen.

5) $\nabla \times (\varphi F) = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \varphi P & \varphi Q & \varphi R \end{vmatrix} =$ se $F = (P, Q, R)$

$= i(\partial_y\varphi\cdot R + \varphi\cdot\partial_y R - \partial_z\varphi\cdot Q - \varphi\partial_z Q)$

$- j(\partial_x\varphi\cdot R + \varphi\cdot\partial_x R - \partial_z\varphi\cdot P - \varphi\cdot\partial_z P) + k(\partial_x\varphi\cdot Q + \varphi\cdot\partial_x Q - \partial_y\varphi\cdot P + \varphi\cdot\partial_y P)$

$$= \begin{vmatrix} i & j & k \\ \partial_x \varphi & \partial_y \varphi & \partial_z \varphi \\ P & Q & R \end{vmatrix} + \varphi \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$= \nabla\varphi \times F + \varphi \cdot \nabla \times F$; cioè $\boxed{\nabla \times (\varphi F) = \nabla\varphi \times F + \varphi \cdot \nabla \times F}$

(6) $f(x,y) = (x^2-25)\cdot(9x^2-25y^2)+15$

$$\begin{cases} f_x(x,y) = 2x(9x^2-25y^2)+(x^2-25)\cdot 18x = 2x(18x^2-25y^2-25\cdot 9 \\ f_y(x,y) = (x^2-25)(-50\cdot y) \end{cases}$$

$$\nabla f(x,y)=0 \iff \begin{cases} y=0 \\ 2x\,[9x^2+(x^2-25)9]=0 \end{cases} \text{ o } \begin{cases} x^2-25=0 \\ 9x^2-25y^2=0 \end{cases}$$

$$\iff \begin{cases} y=0 \\ x=0 \end{cases} \text{ o } \begin{cases} y=0 \\ x^2=\frac{25}{2} \end{cases} \text{ o } \begin{cases} x^2=25 \\ y^2=9 \end{cases}$$

P.ti critici: $\boxed{(0,0);\ \left(\pm\frac{5}{\sqrt{2}},0\right);\ (\pm 5, \pm 3)}$

$$\text{Hess} f(x,y) = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2\cdot(18x^2-25y^2-225)+72x^2 & -100xy \\ -100xy & -50(x^2-25) \end{pmatrix}$$

$\text{Hess} f(0,0) = \begin{pmatrix} - & 0 \\ 0 & + \end{pmatrix}$ non def. $\Rightarrow$ selle

$\text{Hess} f\left(\pm\frac{5}{\sqrt{2}},0\right) = \begin{pmatrix} + & 0 \\ 0 & + \end{pmatrix}$ def. pos. $\Rightarrow$ p.ti min. rel.

$\text{Hess} f(\pm 5, \pm 3) = \begin{pmatrix} \# & \#\neq 0 \\ \#\neq 0 & 0 \end{pmatrix}$ non def. $\Rightarrow$ selle.

$(0,0);\ (\pm 5, \pm 3)$: 5 p.ti di selle

$\pm\left(\frac{5}{\sqrt{2}},0\right)$: p.ti di min. rel.