Prova scritta di Analisi Matematica II (24/1/2013)

Nome..Cognome... Matricola................
Prova orale: inizio appello/fine appello (cancellare la voce che non interessa),
non nella mezza giornata di...

(1) **[14 pti]** Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \{(x,y,z) : \ z^2 \cdot (x^2+y^2) \leq 1, \ 0 \leq z \leq 2, \ x^2+y^2 \leq 4, \ y \leq 0 \}$.
(1.1) Fare un disegno *qualitativo* di Ω.

(1.2) Parametrizzare $\partial\Omega$ e dire se le parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$. (Nella formula devono apparire, magari iterati, solo integrali di una variabile).

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x,y,z) = (x,z,y)$.

(1.5) . Sia $\Sigma = \{(x,y,z) : \ z^2 \cdot (x^2+y^2) = 1, \ 0 \leq z \leq 2, \ x^2+y^2 \leq 4 \}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la normale ν a Σ (ν essendo la normale di cui al punto (1.2)).

(1.6) Calcolare $\iint_\Sigma (\nabla \times F) \cdot \mu d\sigma$, con la stessa F di (1.4).

(2) [2 pti] Dire per quale valore del parametro $\alpha \in \mathbb{R}$ il campo $F_\alpha : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ è chiuso, dove

$$F_\alpha(x,y) = \left(\frac{\alpha x}{x^2+y^2}, \frac{y}{x^2+y^2} \right).$$

Dire se il campo trovato è F è esatto.

(3) [5 pti] Sia $A = \{(x,y) : \ 1 \le x^2+y^2 \le 4, \ x \ge y\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A \log(x^2+y^2) dxdy.$$

(4) **[3 pti]** Risolvere il problema di Cauchy

$$\begin{cases} \dot{x} = \frac{\cos(t)}{2x} \\ x(0) = -\frac{1}{\sqrt{2}} \end{cases}$$

e determinare il dominio della soluzione.

(5) **[2 pti]** Sia $\alpha \in C^1(\mathbb{R}^2, \mathbb{R})$ e si definisca $f : \mathbb{R}^3 \to \mathbb{R}$,

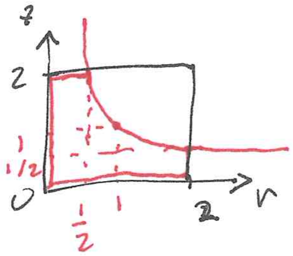
$$f(x, y, z) = x \cdot \alpha(2x^2 + \sin(yz), \cos(yz)).$$

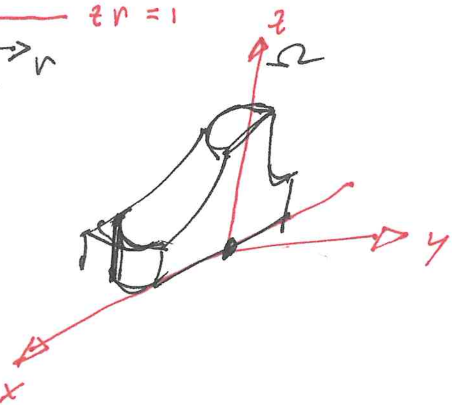
Calcolare $\nabla f(x_0, y_0, z_0)$.

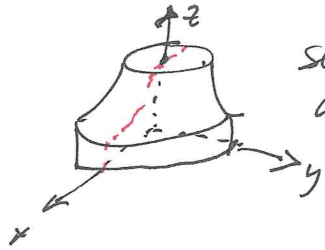
(6) **[4 pti]** Classificare i punti critici di $f(x, y) = (x + y - 4) \cdot (x - y + 4)(y^2 - 2y)$.

(1) $\Omega = \{(x,y,z): z^2 \cdot (x^2+y^2) \le 1;\ 0 \le z \le 2;\ x^2+y^2 \le 4;\ y \le 0\}$. Posti $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $|\theta| \le$ … $r \ge$ …

$(x,y,z) \in \Omega \iff 0 \le z \le 2;\ 0 \le r \le 2;\ zr \le 1;\ -\pi \le \theta \le 0$.

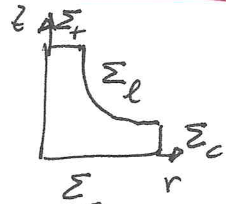






senza la condizione $y < 0$

Parametrizzazioni:



manca la faccia $\Sigma_y = \{(x,y,z) \in \partial\Omega : y = 0\}$

(i) $\Sigma_+ = \{(x,y,2): x^2+y^2 \le 1/4\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_+ = \{(r,\theta): 0 \le r \le 1/2;\ -\pi \le \theta \le 0\} \xrightarrow{\Phi_+} \mathbb{R}^3$ $\quad \Phi_+(A_+) = \Sigma_+$

$\Phi_+(r,\theta) = (r\cos\theta, r\sin\theta, 2)$; $\partial_r\Phi_+ \times \partial_\theta\Phi_+ = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0,0,r)$: compatibile.

(ii) $\Sigma_- = \{(x,y,0): x^2+y^2 \le 4\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_- = \{(r,\theta): 0 \le r \le 2;\ -\pi \le \theta \le 0\} \xrightarrow{\Phi_-} \mathbb{R}^3$ $\quad \Phi_-(A_-) = \Sigma_-$

$\Phi_-(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ $\quad \partial_r\Phi_- \times \partial_\theta\Phi_- = (0,0,r)$: non compatibile

(iii) $\Sigma_c = \{(x,y,z): x^2+y^2 = 4;\ 0 \le z \le 2\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_c = \{(\theta,z): -\pi \le \theta \le 0;\ 0 \le z \le 2\} \xrightarrow{\Phi_c} \mathbb{R}^3$ $\quad \Phi_c(A_c) = \Sigma_c$

$\Phi_c(\theta,z) = (2\cos\theta, 2\sin\theta, z)$ $\quad \partial_\theta\Phi_c \times \partial_z\Phi_c = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta, 2\sin\theta, 0) = 2\cdot(\cos\theta, \sin\theta, 0)$: compatibile.

(iv) $\Sigma_\ell = \{(x,y,z): (x^2+y^2)\cdot z^2 = 1;\ \frac{1}{2} \le z \le 2;\ \frac{1}{4} \le x^2+y^2 \le 4\}$

$\mathbb{R}^2 \supseteq A_\ell = \{(r,\theta): \frac{1}{2} \le r \le 2;\ -\pi \le \theta \le 0\} \xrightarrow{\Phi_\ell} \mathbb{R}^3$ $\quad \Phi_\ell(A_\ell) = \Sigma_\ell$

$\Phi_\ell(r,\theta) = (r\cos\theta, r\sin\theta, 1/r)$

$\partial_r\Phi_\ell \times \partial_\theta\Phi_\ell = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -\frac{1}{r^2} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \left(-\frac{\cos\theta}{r}, -\frac{\sin\theta}{r}, r\right)$

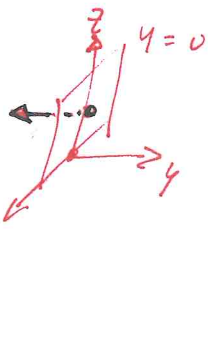
compatibile

$\Sigma_y = \{(x,0,z): z^2x^2 \le 1;\ 0 \le z \le 2;\ x^2 \le 4\}$

$\mathbb{R}^2 \supseteq A_y = \{(x,z): z^2x^2 \le 1;\ 0 \le z \le 2;\ x^2 \le 4\} \xrightarrow{\Phi_y} \mathbb{R}^3$ $\quad \Phi_y(A_y) = \Sigma_y$

$\Phi_y(x,z) = (x,0,z)$ $\quad \partial_x\Phi_y \times \partial_z\Phi_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = (0,-1,0)$

non compatibile.



Per il flusso utilizzo il T. della divergenza.

$(x,y,z)\in\Omega \iff 0\le z\le 1/2$ e $0\le r\le 2$ e $-\pi\le\theta\le 0$

oppure $1/2\le z\le 2$ e $0\le r\le 1/z$ e $-\pi\le\theta\le 0$:

$$\iint_{\partial\Omega} F\cdot\nu\,d\sigma = \iiint_{\Omega} \operatorname{div}F(x,y,z)\,dx\,dy\,dz = \int_0^{1/2} dz\int_0^2 dr\int_{-\pi}^0 d\theta\,.\operatorname{div}F(r\cos\theta, r\sin\theta, z).r$$

$$+\int_{1/2}^2 dz\int_0^{1/z} dr\int_{-\pi}^0 d\theta\,.\operatorname{div}F(r\cos\theta, r\sin\theta, z).r$$

Se $F(x,y,z)=(x,y,z)$, $\operatorname{div}F=3$ e $\iint_{\partial\Omega} F\cdot\nu\,d\sigma =$

$$=3\pi\cdot\frac{1}{2}\cdot\left(\frac{r^2}{2}\right)_0^2+3\pi\cdot\int_{1/2}^2\left(\frac{r^2}{2}\right)_0^{1/z}dz = 3\pi+3\pi\cdot\int_{1/2}^2\frac{dz}{2z^2}$$

$$=3\pi+\frac{3}{2}\pi\cdot\left(-\frac{1}{z}\right)_{1/2}^2 = 3\pi+\frac{3}{2}\pi\cdot\left(-\frac{1}{2}+2\right)=\pi\cdot\left(3+\frac{9}{4}\right)=\frac{21}{4}\pi$$

In (1.5) Σ è come Σ_e, ma senza restringere a $y\le 0$:

$\Gamma_+ = \{(x,y,z): x^2+y^2=\frac{1}{4};\ z=2\}$

$[-\pi,\pi]\xrightarrow{\gamma_+}\mathbb{R}^3$ $\quad\gamma_+(\theta)=\left(\frac{\cos\theta}{2},\frac{\sin\theta}{2},2\right)$ $\quad\gamma_+([-\pi,\pi])=\Gamma_+$

$\Gamma_- = \{(x,y,z): x^2+y^2=4;\ z=1/2\}$

$[-\pi,\pi]\xrightarrow{\gamma_-}\mathbb{R}^3$ $\quad\gamma_-(\theta)=\left(2\cos\theta, 2\sin\theta,\frac{1}{2}\right)$ $\quad\gamma_-([-\pi,\pi])=\Gamma_-$

$\partial\Sigma=\Gamma_+\cup\Gamma_-$; γ_+ non è comp. con ν, γ_- è comp. con ν.

(1.6) ~~[illegible]~~ Uso Stokes: $\iint_{\Sigma}(\nabla\times F)\cdot\nu\,d\sigma = -\int_{\gamma_+}F(s)\cdot ds+\int_{\gamma_-}F(s)\cdot ds$

$$=-\int_0^{2\pi}\left(\frac{\cos\theta}{2},2,\frac{\sin\theta}{2}\right)\cdot\left(-\frac{\sin\theta}{2},\frac{\cos\theta}{2},0\right)d\theta+\int_0^{2\pi}\left(2\cos\theta,\frac{1}{2},2\sin\theta\right)\cdot(-2\sin\theta, 2\cos\theta, 0)\,d\theta$$

$$=\int_0^{2\pi}\left(\frac{\cos\theta\cdot\sin\theta}{4}+\cos\theta-4\sin\theta\cos\theta+\cos\theta\right)d\theta=0$$

per periodicità e cancellazioni.

(2) $\partial_y F_\alpha = \frac{-2\alpha x y}{(x^2+y^2)^2} \overset{?}{=} \partial_x F_\alpha = \frac{-2yx}{(x^2+y^2)^2} \iff \alpha = 1$.

$F = F_1$ è chiuso, $F(x,y) = (x,y) \cdot \frac{1}{x^2+y^2}$ è radiale,
$F: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}^2$ quindi è esatto.

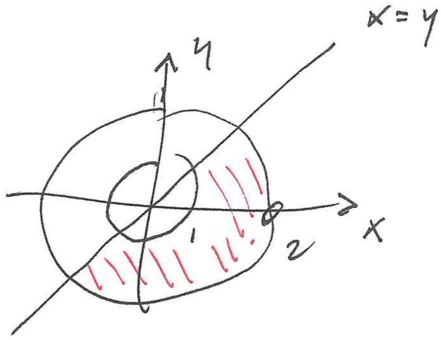
Se $\Phi(x,y) = \varphi(x^2+y^2)$ allora $\nabla\Phi(x,y) = (x,y)\cdot(2\varphi'(x^2+y^2), 2\varphi'(x^2+y^2)$

Voglio $2\varphi'(t) = \frac{1}{t}$ $\quad \varphi(t) = \frac{1}{2}\log t + c$

$\Phi(x,y) = \frac{1}{2}\log(x^2+y^2) + c$, $\quad \Phi: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$

esprime i potenziali di F.

(3)



$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$: $r \geq 0$, $|\theta| \leq \pi$

$(x,y) \in A \iff 1 \leq r \leq 2$, $-\frac{3}{4}\pi \leq \theta \leq \pi/4$

$$\iint_A \log(x^2+y^2)\,dx\,dy = \int_1^2 r\,dr \int_{-3\pi/4}^{\pi/4} d\theta \cdot \log(r^2) = \frac{\pi}{2} \cdot \int_1^4 \log(t)\,dt \qquad t = r^2$$

$$= \frac{\pi}{2} \cdot \left[(t\log t)_1^4 - \int_1^4 \frac{t}{t}\,dt \right] = \frac{\pi}{4}(4\log 4 - 3)$$

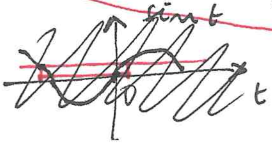
(4)

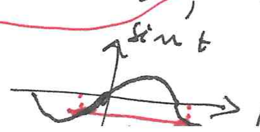
$2x\dot{x}(t) = \cos(t) \iff \begin{cases} \int_0^t 2x\dot{x}(s)\,ds = \int_{x(0)}^{x(t)} 2y\,dy = (y^2)_{x(0)}^{x(t)} = x(t)^2 - x(0)^2 \\ \int_0^t \cos(s)\,ds = (+\sin(s))_0^t = +\sin(t) \end{cases}$

cioè $x(t)^2 = x(0)^2 + \sin(t) = \frac{1}{2} + \sin(t)$

$x(t) = \pm\sqrt{\frac{1}{2} + \sin t}$, ma $x(0) = -\frac{1}{\sqrt{2}}$, allora

$x(t) = -\sqrt{\frac{1}{2} + \sin t}$, Dominio(x) $= (-\pi/6, \pi + \pi/6)$





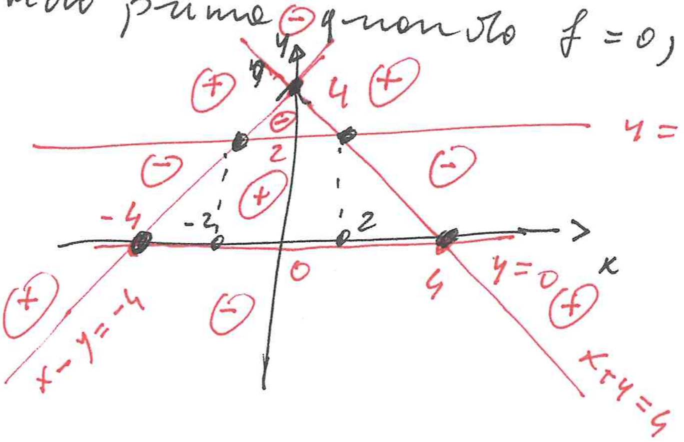
[5] Sia $\alpha = \alpha(v, w)$:

$f_x(x,y,z) = \alpha(2x^2+\sin(yz), \cos(yz)) + x\cdot\alpha_v(2x^2+\sin(yz), \cos(yz))\cdot 2x$

$f_y(x,y,z) = xz\cdot\alpha_v(2x^2+\sin(yz), \cos(yz))\cdot\cos(yz) - xz\cdot\alpha_w(2x^2+\sin(yz), \cos(yz))\cdot\sin(yz)$

$f_z(x,y,z) = xy\cdot\alpha_v(2x^2+\sin(yz), \cos(yz))\cdot\cos(yz) - xy\cdot\alpha_w(2x^2+\sin(yz), \cos(yz))\cdot\sin(yz)$

(6) Vedo prima quando $f = 0, >0, <0$.



Vedo punti di sella: $(0,4), (\pm 4,0), (\pm 2,2)$

Ci saranno almeno un p.to max. rel e un p.to min. rel.: almeno 7 p.ti critici.

$f_x = (x-y+4)(y^2-2y) + (x+y-4)(y^2-2y) = 2x\cdot(y^2-2y)$

$f_y = (x-y+4)(y^2-2y) - (x+y-4)(y^2-2y) + (x+y-4)(x-y+4)\cdot(2y-2)$

$f_x = 0 \iff x = 0$ o $y = 0$ o $y = 2$.

(A) $\begin{cases} y = 0 \\ -2(x-4)(x+4) = 0 \end{cases}$ $(\pm 4, 0) \to$ p.ti di sella

(B) $\begin{cases} y = 2 \\ 2(x-2)(x+2) = 0 \end{cases}$ $(\pm 2, 0) \to$ p.ti di sella

(C) $\begin{cases} x = 0 \\ (4-y)(y^2-2y) - (y-4)(y^2-2y) + (y-4)(4-y)(2y-2) = 0 \end{cases}$

$0 = (4-y)\cdot[2(y^2-2y) + 2\cdot(y-4)(y-1)] = 2\cdot(4-y)\cdot(2y^2-7y+4)$

$(0,4) \to$ p.to di sella.

$y = \frac{7 \pm \sqrt{49-32}}{4} = \frac{7 \pm \sqrt{17}}{4}$: $\left(0, \frac{7+\sqrt{17}}{4}\right) \to$ p.to min. rel.

$\left(0, \frac{7-\sqrt{17}}{4}\right) \to$ p.to Max. rel.