Prova scritta di Analisi Matematica II (12/2/2013)

Nome..Cognome.. Matricola................

Prova orale: inizio appello/fine appello (cancellare se non interessa),

(1) **[14 pti]** Sia $\Omega \subset \mathbb{R}^3$ l'insieme $\Omega = \left\{ (x, y, z) : \sqrt{\frac{x^2}{4} + \frac{y^2}{9}} \leq \cos(z) + 2, \ |z| \leq 2\pi \right\}$.

(1.1) Fare un disegno *qualitativo* di Ω.

(1.2) Parametrizzare $\partial\Omega$ e dire se le parametrizzazioni scelte sono o meno compatibili con il campo ν normale a $\partial\Omega$ esternamente a Ω.

(1.3) Sia $F \in C^1(\Omega, \mathbb{R}^3)$ un campo vettoriale. Scrivere *una* formula esplicita che dia il flusso $\iint_{\partial\Omega} F \cdot \nu d\sigma$ di F attraverso $\partial\Omega$. (Nella formula devono apparire, magari iterati, solo integrali di una variabile).

(1.4) Calcolare il flusso di cui al punto (1.4) quando $F(x, y, z) = (y, -x, z)$.

(1.5) . Sia $\Sigma = \left\{ (x, y, z) : \sqrt{\frac{x^2}{4} + \frac{y^2}{9}} = \cos(z) + 2, \ |z| \leq 2\pi \right\}$. Parametrizzare $\partial\Sigma$ e dire se le parametrizzazioni scelte sono compatibili con la normale ν a Σ (ν essendo la normale di cui al punto (1.2)).

(1.6) Calcolare $\iint_\Sigma (\nabla \times F) \cdot \nu d\sigma$, con la stessa F di (1.4).

(2) [2 pti] Dire per quale valore del parametro $\alpha \in \mathbb{R}$ il campo $F_\alpha : \{(x,y) \in \mathbb{R}^2 : \ x > 0\} \to \mathbb{R}^2$ è chiuso, dove

$$F_\alpha(x,y) = \left(\frac{\alpha y}{\sqrt{x + \sqrt{x^2+y^2}}}, \sqrt{x + \sqrt{x^2+y^2}} \right).$$

Dire se F è esatto.

(3) [5 pti] Sia $A = \left\{(x,y) : \ \frac{x^2}{4} + \frac{y^2}{9} \leq 4, \ x \geq 0, \ y \geq 0\right\} \subset \mathbb{R}^2$. Disegnare A e calcolare

$$\iint_A \frac{\sqrt{\frac{x^2}{4} + \frac{y^2}{9}}}{\frac{x^2}{4} + \frac{y^2}{9} + 1} dxdy.$$

(4) [3 pti] Trovare l'integrale generale di

$$y'' + 4y = e^{2x}$$

(5) [2 pti] Sia $\alpha,\ \beta \in C^1(\mathbb{R}^2, \mathbb{R})$ e si definisca la funzione $h : \mathbb{R}^2 \to \mathbb{R}$,

$$h(x, y) = \int_{\alpha(x,y)}^{\beta(x,y)} \log(t^2 + 1)dt.$$

Calcolare il gradiente $\nabla h(x, y)$ di h in (x, y).

(6) [4 pti] Classificare i punti critici di $f(x, y) = \left(\frac{x^2}{4} - \frac{y^2}{9} + 1\right) \cdot \left(\frac{x^2}{4} - \frac{y^2}{36}\right)$.

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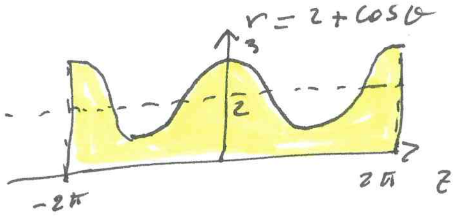
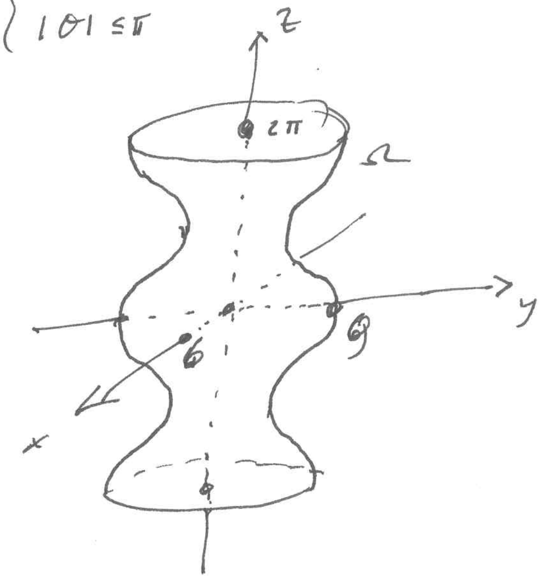
(1.1) $\Omega = \{(x,y,z) : \sqrt{\frac{x^2}{4}+\frac{y^2}{9}} \le \cos(z)+2;\ \ |z| \le 2\pi\}$

$$\begin{cases} \frac{x}{2} = r\cos\theta \\ \frac{y}{3} = r\sin\theta \\ z = z \end{cases} \text{con} \begin{cases} r \ge 0 \\ |\theta| \le \pi \\ z \in \mathbb{R} \end{cases} : (x,y,z) \in \Omega \iff \begin{cases} 0 \le r \le \cos(z)+2 \\ |z| \le 2\pi \\ |\theta| \le \pi \end{cases}$$





Per ulteriori calcoli mi servirà:

$$dx\,dy\,dz = 6\,r\,dr\,d\theta\,dz$$

(1.2) $\partial\Omega = \Sigma_+ \cup \Sigma_- \cup \Sigma_\ell$

$\Sigma_+ = \{(x,y,z) : z = 2\pi,\ \ x^2+y^2 \le 9\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_+ = \{(x,y) : x^2+y^2 \le 9\} \xrightarrow{\Phi_+} \mathbb{R}^3 \qquad \Phi_+(A_+) = \Sigma_+,\ \Phi_+(x,y) = (x,y,2\pi)$

$(\partial_x\Phi_+ \times \partial_y\Phi_+)(x,y) = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$: compatibile con ν.

$\Sigma_- = \{(x,y,z) : z = -2\pi,\ x^2+y^2 \le 9\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_- = A_+ \xrightarrow{\Phi_-} \mathbb{R}^3,\ \ \Phi_-(A_-) = \Sigma_-;\ \ \Phi_-(x,y) = (x,y,-2\pi)$

$(\partial_x\Phi_- \times \partial_y\Phi_-)(x,y) = (\partial_x\Phi_+ \times \partial_y\Phi_+)(x,y) = (0,0,1)$: non compatibile con ν.

$\Sigma_\ell = \{(x,y,z) : |z| \le 2\pi;\ \ \sqrt{\frac{x^2}{4}+\frac{y^2}{9}} = \cos(z)+2\} \subseteq \mathbb{R}^3$

$\mathbb{R}^2 \supseteq A_\ell = \{(\theta,z) : |\theta| \le 2\pi,\ |z| \le 2\pi\} = [-\pi,\pi]\times[-2\pi,2\pi] \xrightarrow{\Phi_\ell} \mathbb{R}^3;\ \ \Phi_\ell(A_\ell) = \Sigma_\ell$

$\Phi_\ell(\theta,z) = (2(\cos(z)+2)\cos\theta;\ 3(\cos(z)+2)\sin\theta;\ z)$

$$(\partial_\theta\Phi_\ell \times \partial_z\Phi_\ell)(\theta,z) = \begin{vmatrix} i & j & k \\ -2(\cos(z)+2)\sin\theta & 3(\cos(z)+2)\cdot\cos\theta & 0 \\ -2\sin(z)\cos\theta & -3\sin(z)\sin\theta & 1 \end{vmatrix}$$

$$= (3(\cos(z)+2)\cos\theta;\ 2(\cos(z)+2)\cdot\sin\theta;\ 6(\cos(z)+2)\sin(z))$$

$\Phi_\ell(0,0) = (6,0,0)$ e $(\partial_\theta\Phi_\ell \times \partial_z\Phi_\ell(0,0)) = (9,0,0)$:

compatibile con ν.

(1.3) Utilizzo le informazioni in (1.1) e T. div.:

$$\boxed{\iint_{\partial\Omega} F\cdot\nu\,d\sigma = \iiint_\Omega \operatorname{div}F(x,y,z)\,dx\,dy\,dz = \int_{-\pi}^{\pi} d\theta \int_{-2\pi}^{2\pi} dz \int_0^{\cos(z)+2} 6r\,dr \cdot \operatorname{div}F(2r\cos\theta, 6r\sin\theta, z)}$$

(1.4) $F(x,y,z) = (y,-x,z) \Rightarrow \operatorname{div} F(x,y,z) = 1$ e per (1.3): $\iint_{\partial\Omega} F\cdot\nu\, d\sigma =$

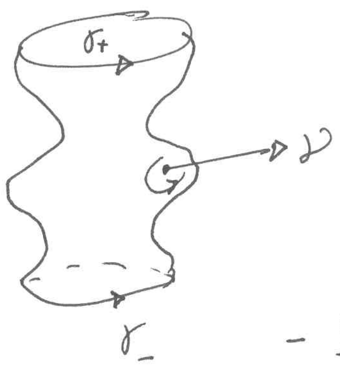
$$= \iiint_\Omega \operatorname{div} F(x,y,z)\,dx\,dy\,dz = \int_{-\pi}^{\pi} d\theta \int_{-2\pi}^{2\pi} dz \int_0^{\cos(z)+2} 6\,r\,dr = 12\cdot\pi\cdot \int_{-2\pi}^{2\pi} \left(\frac{r^2}{2}\right)_0^{\cos(z)+2} dz$$

$$= 6\pi\cdot\int_{-2\pi}^{2\pi} [\cos^2(z) + 4\cdot\cos(z) + 4]\,dz = 6\pi\cdot(2\pi + 0 + 4\cdot 4\pi)$$

$$= 108\cdot\pi^2.$$

(1.5) $\Sigma = \Sigma_e$ e $\partial\Sigma = \Gamma_+ \cup \Gamma_-$; $\Gamma_\pm = \{(x,y,\pm 2\pi) : \frac{x^2}{4}+\frac{y^2}{9} = 9\}$

$[-\pi,\pi] \xrightarrow{\gamma_\pm} \mathbb{R}^3$, $\gamma_\pm([-\pi,\pi]) = \Gamma_\pm$, $\gamma_\pm(\theta) = (6\cos\theta, 9\sin\theta, \pm 2\pi)$



γ_- è compatibile, γ_+ non lo è.

Uso Stokes: $\iint_\Sigma (\nabla\times F)\cdot\nu\, d\sigma = \int_{\gamma_-} F(\xi)\cdot d\xi - \int_{\gamma_+} F(\xi)\cdot d\xi$

$$= \int_{-\pi}^{\pi} (9\sin\theta, -6\cos\theta, -2\pi)\cdot(-6\sin\theta, 9\cos\theta, 0)\,d\theta$$

$$- \int_{-\pi}^{\pi} (9\sin\theta, -6\cos\theta, 2\pi)\cdot(-6\sin\theta, 9\cos\theta, 0)\,d\theta$$

$$= \int_{-\pi}^{\pi} [-54\sin^2\theta - 54\cos^2\theta + 54\sin^2\theta + 54\cos^2\theta]\,d\theta = 0$$

(2) Se $F=(P,Q)$, allora $Q_x(x,y) = \dfrac{1+\frac{2x}{2\sqrt{x^2+y^2}}}{2\sqrt{x+\sqrt{x^2+y^2}}} = \dfrac{(\sqrt{x^2+y^2}+x)}{2\sqrt{x^2+y^2}\cdot\sqrt{x+\sqrt{x^2+y^2}}}$

e $P_y(x,y) = \alpha\cdot \dfrac{\sqrt{x+\sqrt{x^2+y^2}} - y\cdot\dfrac{\frac{2y}{2\sqrt{x^2+y^2}}}{2\sqrt{x+\sqrt{x^2+y^2}}}}{x+\sqrt{x^2+y^2}} =$

$$= \alpha\cdot\frac{2(x+\sqrt{x^2+y^2}) - \frac{y^2}{\sqrt{x^2+y^2}}}{2(\sqrt{x+\sqrt{x^2+y^2}})^3}$$

$$= \alpha\cdot\frac{2x\cdot\sqrt{x^2+y^2}+2x^2+2y^2-y^2}{2\sqrt{x^2+y^2}(\sqrt{x+\sqrt{x^2+y^2}})^3}$$

$$= \alpha\cdot\frac{2x\cdot\sqrt{x^2+y^2}+2x^2+y^2}{2\sqrt{x^2+y^2}\cdot(\sqrt{x+\sqrt{x^2+y^2}})^2}$$

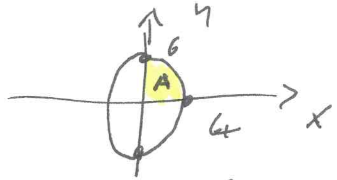
$$P_y = Q_x \iff \alpha\cdot(2x\sqrt{x^2+y^2}+2x^2+y^2) \overset{*}{=} \alpha\cdot(\sqrt{x^2+y^2}+x)^2 = \alpha(x^2+y^2+2\sqrt{x^2+y^2}\cdot x + x^2) = \alpha(2x^2+y^2+2x\sqrt{x^2+y^2}) \iff \alpha = 1$$

Poiché $\{(x,y): x>0\}$ è convesso ///, per $\alpha=1$ ho che F è esatto.

(3) Posti $\begin{cases} \frac{x}{2} = r\cos\theta \\ \frac{y}{3} = r\sin\theta \end{cases}$ con $\begin{array}{l} r \geq 0 \\ |\theta| \leq \pi \end{array}$ ho che $dx\,dy = 6\,r\,dr\,d\theta$

e $(x,y) \in A \iff 0 \leq r \leq 2$

e $0 \leq \theta \leq \pi/2$



$$\text{Integrale} = \int_0^{\pi/2} d\theta \int_0^2 6\,r\,dr \cdot \frac{r}{r^2+1}$$

$$= 3\pi \cdot \int_0^2 \frac{r^2}{r^2+1}\,dr = 3\pi \cdot \int_0^2 \left(1 - \frac{1}{1+r^2}\right) dr = 3\pi \cdot \Big(2r - \operatorname{arctg}(r)\Big)_0^2$$

$$= 3\pi \cdot (2 - \operatorname{arctg}(2))$$

(4) $y'' + 4y = e^{2x}$

$z'' + 4z = 0$

$\lambda^2 + 4 = 0 : \ \lambda = \pm 2i$ $\implies z(x) = A\cdot\cos(2x) + B\cdot\sin(2x)$

Provo con $y(x) = k\cdot e^{2x} \implies y'' + 4y = 4ke^{2x} + 4k\,e^{2x} = 8k\,e^{2x} = e^{2x}$ $\Updownarrow$ $k = 1/8$

Int. gen.: $y(x) = A\cdot\cos(2x) + B\cdot\sin(2x) + \frac{1}{8}e^{2x}$

$A, B \in \mathbb{R}, \quad y : \mathbb{R} \to \mathbb{R}.$

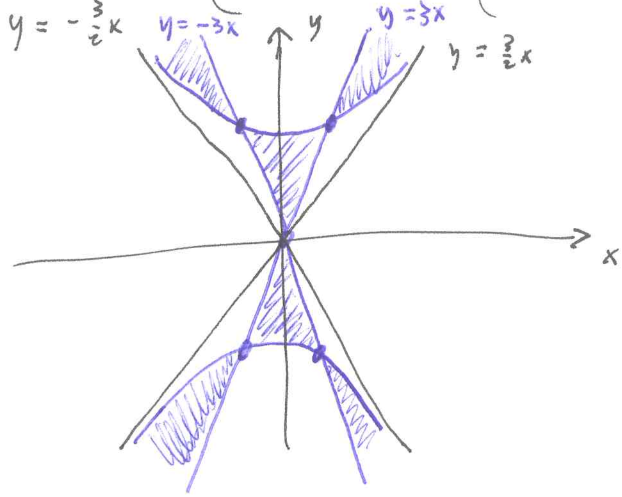
(5) $h(x,y) = \int_0^{\beta(x,y)} \log(t^2+1)\,dt - \int_0^{\alpha(x,y)} \log(t^2+1)\,dt.$ Uso T.F.C.I.

e derivata di composizioni:

$\partial_x h(x,y) = \log(\beta(x,y)^2+1)\cdot\partial_x\beta(x,y) - \log(\alpha(x,y)^2+1)\cdot\partial_x\alpha(x,y)$

$\partial_y h(x,y) = \log(\beta(x,y)^2+1)\cdot\partial_y\beta(x,y) - \log(\alpha(x,y)^2+1)\cdot\partial_y\alpha(x,y)$

(6) $f(x,y) = \left(\frac{x^2}{4} - \frac{y^2}{9} + 1\right)\cdot\left(\frac{x^2}{4} - \frac{y^2}{36}\right)$



Dal grafico di $f(x,y)=0$ deduco la presenza di 5 punti di sella. Dall'analisi del segno (campiggiate i $f<0$) deduco la presenza di due punti di min. relativo: almeno 7 punti critici.

$$\begin{cases} f_x = \frac{\cancel{x}}{\cancel{2}}\left(\frac{x^2}{4} - \frac{y^2}{36}\right) + \frac{\cancel{x}}{2}\left(\frac{x^2}{4} - \frac{y^2}{9} + 1\right) = \frac{x}{2}\cdot\left(\frac{x^2}{2} - \frac{5}{36}y^2 + 1\right) \\ f_y = -\frac{2}{9}y\left(\frac{x^2}{4} - \frac{y^2}{36}\right) - \frac{2}{36}y\left(\frac{x^2}{4} - \frac{y^2}{9} + 1\right) = -\frac{y}{18}\left(\frac{5}{4}x^2 - \frac{2}{9}y^2 + 1\right) \end{cases}$$

$\nabla f(x,y) = 0 \iff \begin{cases} x = 0 \\ y = 0 \text{ o } y = \pm\frac{3}{\sqrt{2}} \end{cases}$ oppure $\begin{cases} \frac{x^2}{4} - \frac{y^2}{36} = 0 \\ \frac{x^2}{4} - \frac{y^2}{9} + 1 = 0 \end{cases}$

$(0,0)$, $\left(0, \pm\frac{3}{\sqrt{2}}\right)$ oppure $\begin{cases} x^2 = \frac{y^2}{9} \\ y^2\left(\frac{1}{36} - \frac{1}{9}\right) + 1 = 0 \end{cases}$ $\begin{cases} x^2 = y^2/9 \\ y^2 = \frac{1}{12} \end{cases}$ $\begin{cases} x = \pm\frac{1}{6\sqrt{3}} \\ y = \pm\frac{1}{2\sqrt{3}} \end{cases}$

$(0,0)$: sella; $\left(0, \pm\frac{3}{\sqrt{2}}\right)$: p.ti min. rel.

$\left(\pm\frac{1}{6\sqrt{3}}, \pm\frac{1}{2\sqrt{3}}\right) \to$ 4 p.ti di sella.

(c)