§1.3 Line integrals in the complex plane.

We start by reviewing the basic definitions and formulas related to line integrals in the plane.

A (piecewise) smooth curve in $\Omega \subseteq \mathbb{R}^2$ (open) is a map $I \xrightarrow{\gamma} \Omega$ from some interval $I \subseteq \mathbb{R}$ and such that $\dot\gamma(t) = \frac{d\gamma}{dt}(t)$ exists at all points $t \in I$, but possibly finitely many of them.

The curve is regular if $\gamma \in C^1(I, \mathbb{R}^2)$ and $\dot\gamma(t) \neq 0$ for $t \in I$.

Let $F \in C(\Omega, \mathbb{R}^2)$ be a continuous field:

$$\int_\gamma F(z) \cdot dz \overset{\text{Df.}}{=} \int_I F(\gamma(t)) \cdot \dot\gamma(t)\, dt$$

is the integral (or work) of F along γ.

A field $F \in C^1(\Omega, \mathbb{R}^2)$ is closed if $F = (P, Q)$ with $\frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y) \quad \forall (x,y) \in \Omega$.

$F \in C(\Omega, \mathbb{R}^2)$ ~~...~~ is exact if $\exists \varphi \in C^1(\Omega, \mathbb{R})$ s.t.

$$\nabla \varphi = F,$$

i.e. $\partial_x \varphi = P$ and $\partial_y \varphi = Q$.

Basic fact. If $F \in C^1(\Omega, \mathbb{R}^2)$ is exact, then it is closed:

$$\partial_y P = \partial_y(\partial_x \varphi) = \partial_{yx} \varphi = \partial_{xy} \varphi = \partial_x(\partial_y \varphi) = \partial_x Q.$$

Theorem. If $F \in C^1(\Omega, \mathbb{R}^2)$ is closed and Ω is simply connected, then F is exact.

$\gamma : [a,b] \to \Omega$
$\gamma(a) = \gamma(b)$

Here, Ω is simply connected if regular, closed curves in Ω can be shrunk to a point while remaining in Ω:

Ω_1 it's not simply connected

Ω_2 it's simply connected

Example: $F(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right)$; $F: \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2$

F is closed: $\partial_y P = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$$\parallel$$

$\partial_x Q = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

but F is not exact, as we shall see shortly.

Theorem. $F \in C^1(\Omega, \mathbb{R}^2)$ is exact $\Longleftrightarrow$ for all closed regular curves γ in Ω we have $\int_\gamma F(z) \cdot dz = 0$.

Back to the example. Let $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$.

Then $\int_\gamma F(z) \cdot dz = \int_0^{2\pi} \left(\frac{-\sin t}{1} \cdot d\cos t + \frac{\cos t}{1} \cdot d\sin t \right)$

$= \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi \neq 0$: F is not exact.

Remark. Let $f = u + iv$ be holomorphic in Ω.

Then, (v, u) and $(u, -v)$ are closed. If Ω is simply connected, they are exact.

In fact $v_y = u_x$ and $u_y = -v_x$ is just C.R.

Complex line integral. If $g = p + iq \in C(\Omega, \mathbb{C})$,

$$\int_\gamma g(z)\, dz := \int_I g(\gamma(t))\, \dot\gamma(t)\, dt$$

is the line integral of g in the complex sense.

Here the product is product in $\mathbb{C}$.

Let's be explicit for $f = u + iv$, $dz = dx + i dy$:

$$\int_\gamma f(z)\, dz = \int_I (u+iv)(dx + i dy) = \int_I \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy) \right]$$

$$= \int_I \left[(u, -v) \cdot (dx, dy) + i\, (v, u) \cdot (dx, dy) \right]$$

Theorem. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected. Then $f \in C^1(\Omega, \mathbb{C})$ is holomorphic iff

$\forall \gamma$ curve closed in $\Omega \Rightarrow \int_\gamma f(z)\,dz = 0$.

Proof. We prove the theorem of Cauchy under the assumption that $f \in C^1(\Omega, \mathbb{C})$, reducing it to a calculation with forms. By the calculation at the end of the previous page,

$$\int_\gamma f(z)\,dz = \int_\gamma (u+iv)(dx+i\,dy) = \int_{I=[a,b]} [(u,-v)\cdot(dx,dy) + i(v,u)\cdot(dx,dy)]$$

Thus, $\int_\gamma f(z)\,dz = 0$ $\forall \gamma$ closed in Ω $\Leftrightarrow$ $\int_I (u,-v)\cdot(dx,dy) = 0$ and $\int_I (v,u)\cdot(dx,dy) = 0$ $\forall \gamma$ closed in Ω

Since Ω is simply connected, the latter holds $\Leftrightarrow$ $(u,-v)$ and (v,u) are exact $\Leftrightarrow$ $(u,-v)$ and (v,u) are closed

$\Leftrightarrow$ $u_y = -v_x$ and $v_y = u_x$ i.e. if and only if the Cauchy-Riemann equations hold, if and only if f is holomorphic. ∎

Thm. Variation on this theorem. Let $\Omega \subseteq \mathbb{C}$ be open. $f \in C^1(\Omega, \mathbb{C})$ is holomorphic if and only if

$\forall \gamma$ closed curve in Ω such that γ can be shrunk to a point in Ω $\Rightarrow \int_\gamma f(z)\,dz = 0$.

The proof is the same.

Thm. Another variation. Let $\Omega \subseteq \mathbb{C}$ be open and simply connected and let f be holomorphic in Ω. Then, $\exists g$ holomorphic in Ω such that $g' = f$ in Ω.

If $z_0 \in \Omega$ is fixed,

$$g(z) - g(z_0) = \int_{z_0}^{z} f(\zeta)\,d\zeta,$$

the integral being performed on any curve γ joining z_0 and z in Ω.

Proof. Let $f = u + iv$ and recall that $(u,-v)$ and (v,u) are exact in Ω. Hence, there are $\phi \in C^1(\Omega,\mathbb{R})$ and $\psi \in C^1(\Omega,\mathbb{R})$ s.t. $\nabla\phi = (u,-v)$ and $\nabla\psi = (v,u)$.

i.e. $\phi_x = u = \psi_y$ and $\phi_y = -v = -\psi_x$.

But these are exactly C.R. equations for $g = \phi + i\psi$, which is then holomorphic.

Now, on the one hand $g(z) - g(z_0) =$

$$= \phi(z) - \phi(z_0) + i\left[\psi(z) - \psi(z_0)\right] = \int_\gamma \nabla\phi(\zeta)\cdot(dx,dy) + i\cdot\int_\gamma \nabla\psi(\zeta)\cdot(dx,dy)$$

$$= \int_\gamma (u,-v)\cdot(dx,dy) + i\cdot\int_\gamma (v,u)\cdot(dx,dy) = \dots = \int_\gamma f(z)\,dz,$$

by previous calculations.

The value of this theorem is mainly indirect: not very often g is explicitly computed by integration. A counterexample (which is very important!!).

$f(z) = \frac{1}{z}$ is holomorphic in $\mathbb{C}\setminus\{0\}$, which is not simply connected. Let's compute for $\gamma(t) = re^{it}$ ($r>0$ fixed, $0 \le t \le 2\pi$) $\int_\gamma f(z)\,dz =$

$$= \int_0^{2\pi} \frac{1}{re^{it}}\, d(re^{it}) = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\, dt = 2\pi i \neq 0:$$

the thesis of the theorem does not hold.

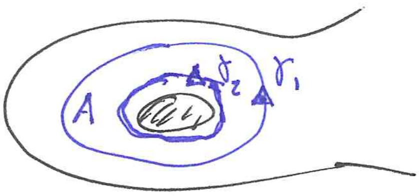
This situation is rather special.

Counter-counterexample. Let $f(z) = \frac{1}{z^n}$ $(n \ge 2)$.

Then, $$\int_\gamma f(z)\,dz = \int_0^{2\pi} \frac{d(re^{it})}{(re^{it})^n} = \frac{i}{r^{n-1}}\cdot \underbrace{\int_0^{2\pi} e^{-i(n-1)t}\,dt}_{=0}$$

Observe the connection with Fourier series. We shall return on this point.

Cauchy's Theorem greatly gains from topology.



A family $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of smooth curves in $\Omega \subseteq \mathbb{C}$ (open) is a <u>boundary</u> if there is an open subset $A \subseteq \bar{A} \subseteq \Omega$ such that the boundary of A ∂A, is parametrized by Γ.

By this we mean that, if $\gamma_j : I_j = [a_j, b_j] \longrightarrow \mathbb{C}$, then (i) γ_j is injective on $[a_j, b_j)$ and $\dot{\gamma}_j(t) \neq 0 \ \forall t \in I_j$

(ii) $\partial A = \bigcup_{j=1}^{n} \gamma_j(I_j)$

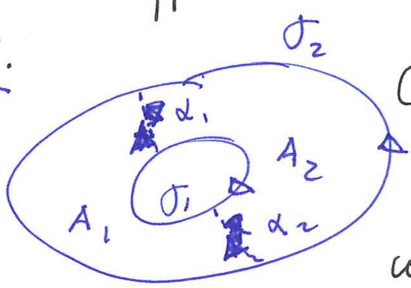
(iii) $\gamma_j(t) = \gamma_i(s)$ for $i \neq j$ or $t \neq s$ $\Rightarrow$ $\gamma_j(t)$ and $\gamma_i(s)$ are extremes of the arcs $\gamma_j(I_j)$ and $\gamma_i(I_i)$.

(iv) A is on the left of each γ_j (see picture).

<u>Theorem</u>. Let $f \in C(\Omega, \mathbb{C})$. Then f is holomorphic if and only if for all boundaries Γ in Ω we have

$$\int_\Gamma f(\zeta)\, d\zeta := \sum_{j=1}^{n} \int_{\gamma_j} f(\zeta)\, d\zeta = 0.$$

<u>Proof</u>.



($\Rightarrow$) We can cut A with curves $\alpha_1, \dots, \alpha_m$ in regions $A_1, \dots, A_k$ which are simply connected. By Cauchy Theorem, if f is holomorphic

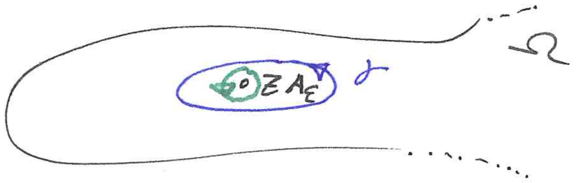
$$\forall i = 1, \dots, k \Rightarrow \int_{\partial A_i} f(\zeta)\, d\zeta = 0.$$

On the other hand, $\int_\Gamma f(\zeta)\, d\zeta = \sum_j \int_{\partial A_j} f(\zeta)\, d\zeta$ because ~~each~~ the integral on each α_ℓ appears twice, with opposite signs.

Cauchy's Reproducing formula. Let γ be a closed curve in Ω without self-intersections (a simple closed curve) and let $f \in Hol(\Omega)$ be holomorphic. Then, for all z in the interior of γ one has

$$\boxed{\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z)}$$

i.e. the values of f inside γ can be reconstructed from the values of f on γ.



Proof. Let $\varepsilon > 0$ be small enough so that $\overline{\Delta}(z,\varepsilon) = \{w \in \mathbb{C} : |w - z| \leq \varepsilon\}$ lies inside γ and let A_ε be the portion of Ω lying between γ and $\partial\Delta(z,\varepsilon) = \{w \in \mathbb{C} : |w-z| \leq \varepsilon\}$. We can apply ~~the~~ Cauchy's Theorem to the function

$$w \longmapsto \frac{f(w)}{w - z},$$

which is holomorphic in an open subset of Ω containing $\overline{A_\varepsilon}$. Since the (oriented) boundary of A_ε consists of γ (anticlockwise) and of $\partial\Delta(z,\varepsilon)$ (clockwise), we have that

$$0 = \frac{1}{2\pi i}\int_{\partial A_\varepsilon} \frac{f(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\partial\Delta(z,\varepsilon)} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z + \varepsilon e^{i\theta})\cdot \varepsilon i e^{i\theta}\,d\theta}{\varepsilon \cdot e^{i\theta}}$$

because $\zeta = z + \varepsilon e^{i\theta}$ $(0 \leq \theta \leq 2\pi)$ parametrizes $\partial\Delta(z,\varepsilon)$ anticlockwise,

$$= \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi}\int_0^{2\pi} f(z + \varepsilon e^{i\theta})\,d\theta$$

$$\xrightarrow[\varepsilon \to 0]{} \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z),$$

then $0 = \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta - f(z)$ as wished ∎

Comment on the limit. The continuity of f is used here: $\forall \eta > 0 \ \exists \varepsilon > 0 : \forall \zeta : |\zeta - z| \le \varepsilon \Rightarrow |f(\zeta) - f(z)| \le \eta$.

Hence,
$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\, d\theta - f(z) \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} [f(z+\varepsilon e^{i\theta}) - f(z)]\, d\theta \right|$$
$$\le \frac{1}{2\pi} \int_0^{2\pi} |f(z+\varepsilon e^{i\theta}) - f(z)|\, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} \eta\, d\theta = \eta$$

i.e. $\forall \eta > 0 \ \exists \varepsilon > 0 : \left| \frac{1}{2\pi} \int_0^{2\pi} f(z + \varepsilon e^{i\theta})\, d\theta - f(z) \right| \le \eta$,

which means that $\lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_0^{2\pi} f(z+\varepsilon e^{i\theta})\, d\theta = f(z)$.

Extensions of the above to the boundary.

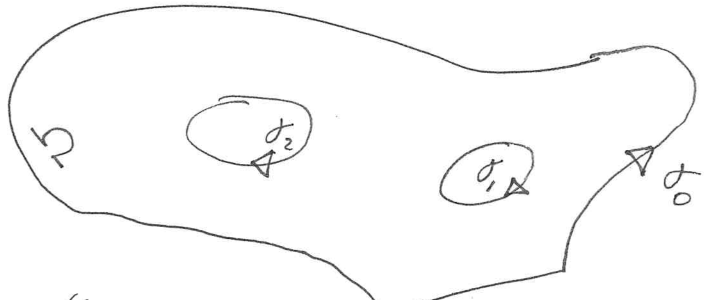
Enhanced Cauchy Theorem. Let Ω be a bounded, open subset of $\mathbb{C}$ and suppose that $\partial\Omega$ can be parametrized as a finite family of smooth simple closed curves $\gamma_0, \gamma_1, \dots, \gamma_n$, with γ_0 separating Ω from ∞.

Let $f \in Hol(\Omega) \cap C(\bar{\Omega})$.

Then

(I) Cauchy Theorem. $\frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta)\, d\zeta = 0$

(II) Cauchy Formula. $\forall z \in \Omega \Rightarrow f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z}\, d\zeta$



The proof relies on passing in the limit on a family of boundaries in Ω approaching $\partial\Omega$. The procedure is intuitive, but carrying it out is lengthy.

Mean Value Theorem. Let $f \in Hol(\Omega)$ and $z_0 \in \Omega$, $r > 0$ such that $\overline{\Delta(z_0,r)} \subseteq \Omega$. Then,

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$$

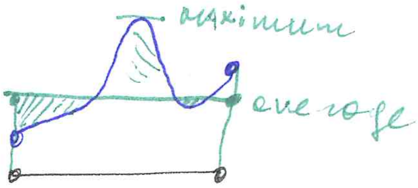
i.e. the value of f at the center of $\overline{\Delta(z_0,r)}$ is the mean of the values of f on it boundary.

Proof.
$$f(z_0) = \frac{1}{2\pi i}\int_{\partial\Delta(z_0,r)} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})\,rie^{i\theta}}{re^{i\theta}}\,d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta \quad \blacksquare$$

Exercise. Show the "thick version" of the M.V.T. If $|A|$ denotes the surface area of $A \subseteq \mathbb{C}$, show that for holomorphic f as in the M.V.P. one has

$$f(z_0) = \frac{1}{|\Delta(z_0,r)|}\int_{\Delta(z_0,r)} f(u+iv)\,du\,dv$$

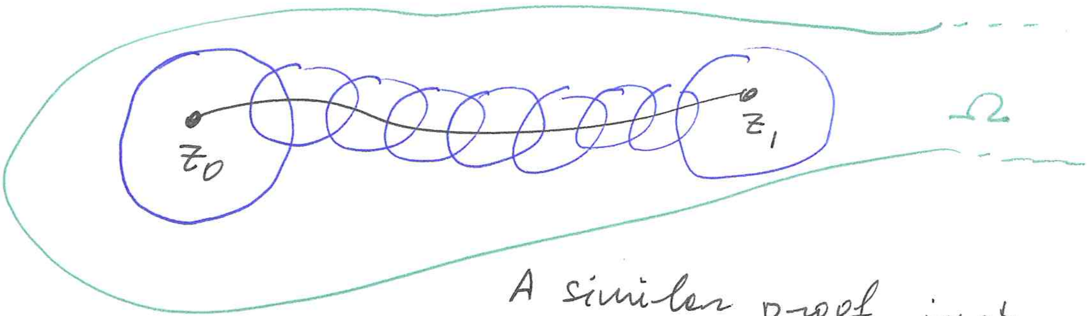
Average value is not best value.



Maximum principle. Let $\Omega \subseteq \mathbb{C}$ be open connected and $f \in Hol(\Omega)$. If $z_0 \in \Omega$ and $|f(z_0)| = \max\{|f(z)| : z \in \Omega\}$, then f is constant in Ω.

Pf. We prove that if $\Delta(z_0,R) \subseteq \Omega$, then f is constant in $\Delta(z_0,R)$.
Suppose f is not constant in $\Delta(z_0,R)$. Then there is $z_0 + re^{i\theta}$ with $0 < r < R$ such that $|f(z_0+re^{i\theta})| < |f(z_0)|$.
Since f is continuous, there is $\delta > 0$ s.t. $|\psi - \theta| \le \delta \Rightarrow$
$$|f(z_0 + re^{i\psi})| \le |f(z_0)| - \delta.$$
By the MVT and the hypothesis,
$$|f(z_0)| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(z_0+re^{i\psi})\,d\psi\right| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z_0+re^{i\psi})|\,d\psi$$
$$= \frac{1}{2\pi}\int_{|\psi-\theta|\le\delta} |f(z_0+re^{i\psi})|\,d\psi + \frac{1}{2\pi}\int_{|\psi-\theta|\ge\delta} |f(z_0+re^{i\psi})|\,d\psi$$
$$\le \frac{1}{2\pi}\,2\delta\big(|f(z_0)| - \delta\big) + \frac{1}{2\pi}(2\pi - 2\delta)\,|f(z_0)|$$
$$= |f(z_0)| - \frac{\delta^2}{\pi},$$ which is absurd.

The full proof can be obtained by joining z_0 and $z_1 \in \Omega$ with a chain of small circles contained in Ω.



A similar proof, just a bit more sophisticated topologically, shows that

Local Maximum Principle. If $\Omega \subseteq \mathbb{C}$ is connected, open, and $f \in Hol(\Omega)$, $z_0 \in \Omega$ and $\exists\, \varepsilon > 0$ s.t.

$$|f(z_0)| \geq |f(z)| \quad \text{whenever } |z - z_0| \leq \varepsilon,$$

Then f is constant in Ω.

~~An immediate consequence of the max. princ.~~

~~Open mapping theorem. If $\Omega \subseteq \mathbb{C}$ is open and connected and $f \in Hol(\Omega)$ is not constant, then $f(\Omega)$ is open.~~

§10. Applications of Cauchy formula to series representation

Geometric series.

(i) $\forall z \in \mathbb{C}$, $|z| < 1 \Rightarrow \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and the convergence is uniform for $|z| \leq r < 1$.

By this we mean that

$$f_N(z) := \sum_{n=0}^{N} z^n \xrightarrow[N \to \infty]{} f(z) = \frac{1}{1-z} \quad \text{uniformly in } |z| \leq r,$$

i.e. $\forall \varepsilon > 0\ \exists\, N_\varepsilon \geq 0 : \forall N \geq N_\varepsilon$ and $\forall z$ s.t. $|z| \leq r$ one has that $|f_N(z) - f(z)| \leq \varepsilon$.

Pf. $f_N(z) = 1 + z + \dots + z^N = \frac{1 - z^{N+1}}{1-z}$, hence

$$|f_N(z) - f(z)| = \left| \frac{(1 - z^{N+1}) - 1}{1-z} \right| \leq \frac{r^{N+1}}{1-r} \xrightarrow[N \to \infty]{} 0 \quad \blacksquare$$

Exercise. Show that convergence is total: there are numbers $a_n \geq 0$ s.t. $\forall z$ s.t. $|z| \leq r$ one has $|z^n| \leq a_n$ and $\sum_{n=0}^{\infty} a_n$ converges.

(ii) Changing variables in (i) we have that if $\zeta, z_0, z \in \mathbb{C}$ and $|z - z_0| < |\zeta - z_0|$, then

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} =$$

$$= \frac{1}{\zeta - z_0} \cdot \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^n} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}}$$

Theorem (Cauchy formula for derivatives).
Let Ω be a bounded, open subset of $\mathbb{C}$ and suppose that $\partial\Omega$ can be parametrized as a finite family of smooth, simple closed curves $\gamma_0, \gamma_1, \dots, \gamma_m$, with γ_0 separating Ω from ∞.

Let $f \in Hol(\Omega) \cap C(\bar{\Omega})$.

Then $\forall m \in \mathbb{N}$ exists $f^{(m)}(z)$ $\forall z \in \Omega$ and

$$f^{(m)}(z) = \frac{m!}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{m+1}} d\zeta$$

Corollary. $\Omega \subseteq \mathbb{C}$ open and $f \in Hol(\Omega) \Rightarrow \forall m \in \mathbb{N}$ $\exists f^{(m)}(z)$, $\forall z \in \Omega$.

In particular, $f \in Hol(\Omega) \Rightarrow f \in C^{\infty}(\Omega)$.

Pf. of the corollary. Let $z \in \Omega$. Consider $\overline{\Delta(z,r)} \subseteq \Omega$ and apply the above theorem to $\overline{\Delta(z,r)}$ instead of Ω. ∎

Pf. of the Theorem. We may take derivatives under the integral sign:

$$f'(z) = \frac{d}{dz}\left[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)} d\zeta\right] = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{d}{dz}\left(\frac{f(\zeta)}{\zeta - z}\right) d\zeta$$

$$= \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$ and, by induction,

$$f^{(m)}(z) = \frac{d}{dz} f^{(m-1)}(z) = \frac{d}{dz}\left[\frac{(m-1)!}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^m} d\zeta\right] =$$

$$= \frac{(m-1)!}{2\pi i} \int_{\partial\Omega} \frac{d}{dz}\left[\frac{f(\zeta)}{(\zeta - z)^m}\right] d\zeta = \frac{(m-1)!}{2\pi i} \int_{\partial\Omega} \frac{m \cdot f(\zeta)}{(\zeta - z)^{m+1}} d\zeta$$

$$= \frac{m!}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z)^{m+1}} d\zeta$$ as wished ∎

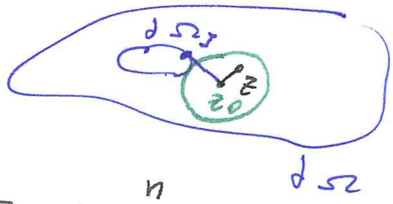
Ask me if you do not trust derivatives under the integral sign.

The Theorem above is pretty strong, but the following is even better!

Theorem (Power expansion of holomorphic functions). Let Ω be a bounded, open subset of $\mathbb{C}$ and suppose that $\partial\Omega$ can be parametrized as ~~the~~ finite ~~union~~ family of smooth, simple closed curves $\gamma_0, \dots, \gamma_{\nu}$; with γ_0 separating Ω from ∞.

Let $f \in Hol(\Omega)$, let $z_0 \in \Omega$ and let $r > 0$ be s.t. $\forall \zeta \in \partial\Omega \Rightarrow |\zeta - z_0| \geq r$. Then,

$\forall z \in \Omega$, $|z - z_0| < r \Rightarrow$



$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \cdot (z - z_0)^n$$

Corollary. If $\Omega \subseteq \mathbb{C}$ is open, $f \in Hol(\Omega)$ and $z_0 \in \Omega$, then $\exists r > 0$ s.t. $\forall z \in \mathbb{C}$, $|z - z_0| < r \Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers.

Obs. Compare this with the definition of Z-transform: all holomorphic functions f are locally of the form $f(z) =$ Z-transform of $\{a_n\}$ computed in $w = \frac{1}{z - z_0}$.

We will see that the viceversa holds as well.

Pf. of the Theorem (assuming we can freely switch integrals and series).

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta - z} d\zeta \overset{\text{by (ii)}}{=} \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta) \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} d\zeta$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \cdot (z - z_0)^n \quad \blacksquare$$

Corollary: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ and $a_n = \frac{f^{(n)}(z_0)}{n!}$