Prova scritta di Analisi Matematica L-B

11 giugno 2011

Nome...**Cognome**.. **Matricola**................

Prova orale: non nel giorno...................................

(1) [**4 pti**] Sia $A = \left\{(x,y) : \frac{x^2}{9} + \frac{y^2}{25} \leq 4,\ \frac{x^2}{9} - \frac{y^2}{25} \geq 1,\ y \geq 0\right\} \subset \mathbb{R}^2$. Calcolare

$$\iint_A y\,dx\,dy.$$

(2) [**8 pti**] Classificare i punti critici di $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy\left(\frac{x}{3} + \frac{y}{5} - 1\right) + 15$.

(3) [**4 pti**] Trovare l'integrale generale di $\ddot{x} + 9\dot{x} = 1 + \sin(3t)$.

(4) [3 pti] Sia $f \in C^1(R^2, R)$, e sia

$$g(r,\theta) = f(r\cos(\theta), r\sin(\theta)).$$

Calcolare $\frac{\partial g}{\partial r}(1, \frac{\pi}{2})$ e $\frac{\partial^2 g}{\partial r^2}(1, \frac{\pi}{2})$.

(5) [5 pti] Siano $\Omega \subset \mathbb{R}^3$ l'insieme

$$\Omega = \left\{(x,y,z) : \, 1 \leq \frac{x^2}{9} + \frac{y^2}{25} \leq 4 - z, \; z \geq 0\right\}.$$

e $f \in C(\Omega, \mathbb{R})$ continua.

Trovare $A \subset \mathbb{R}^2$ e, per $(x,y) \in \mathbb{R}^2$, trovare $\alpha(x,y), \beta(x,y) \in \mathbb{R}$, tali che

$$\iiint_\Omega f(x,y,z)dxdydz = \iint_A \left[\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)dz\right] dxdy$$

(6) [3 pti] Trovare le soluzioni in $\mathbb{C}$ dell'equazione

$$z^4 - 9z^2 + 81 = 0$$

e calcolarne la parte reale.

(7) [3 pti] Trovare i valori di $\gamma \geq 0$ tali che converga la serie:

$$\sum_{n=1}^{\infty} n^{2\gamma} \frac{(1 + 1/n)^\gamma - 1}{n}.$$

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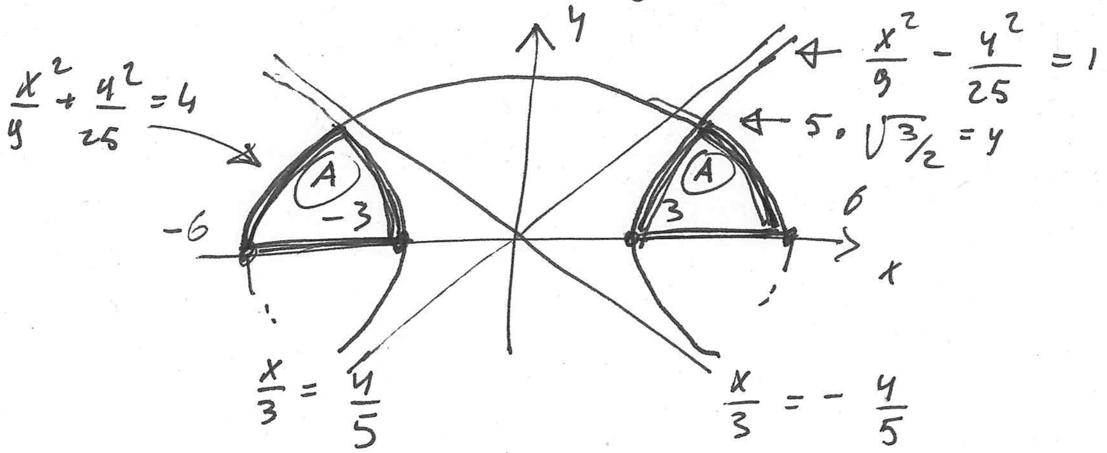
(1) $A = \left\{ \frac{x^2}{9} + \frac{y^2}{25} \le 4;\ \frac{x^2}{9} - \frac{y^2}{25} \ge 1;\ y \ge 0 \right\}$



$$\begin{cases} \frac{x^2}{9} + \frac{y^2}{25} = 4 \\ \frac{x^2}{9} - \frac{y^2}{25} = 1 \end{cases} \qquad \begin{cases} 2 \cdot \frac{x^2}{9} = 5 \\ \dots \end{cases} \qquad \begin{cases} x^2 = \frac{45}{2} \\ y^2 = 25 \cdot \left(4 - \frac{45}{2 \cdot 9}\right) = 25 \cdot \frac{3}{2} \end{cases}$$

$$I = \iint_A y \, dx \, dy = 2 \cdot \int_0^{5\sqrt{3/2}} y \, dy \int_{\sqrt{1 + \frac{y^2}{25}}}^{\sqrt{4 - y^2/25}} dx$$

$$= 2 \cdot \int_0^{5\sqrt{3/2}} y \, dy \cdot \left(\sqrt{4 - y^2/25} - \sqrt{1 + y^2/25} \right)$$

$$= 2 \cdot \left[\frac{(4 - y^2/25)^{3/2}}{3/2 \cdot (-2/25)} - \frac{(1 + y^2/25)^{3/2}}{3/2 \cdot 2/25} \right]_0^{5\sqrt{3/2}}$$

$$= -2 \cdot \frac{25}{3} \cdot \left[(4 - 3/2)^{3/2} - 4^{3/2} - (1 + 3/2)^{3/2} + 1 \right]$$

$$= \frac{2 \cdot 25}{3} \cdot \left[7 + \left(\frac{5}{2}\right)^{3/2} - \left(5/2\right)^{3/2} \right] = \frac{2 \cdot 7 \cdot 25}{3}$$

(2) $f = xy\left(\frac{x}{3}+\frac{y}{5}-1\right)+15$

$f_x = y\left(\frac{x}{3}+\frac{y}{5}-1\right)+xy\,\frac{1}{3} = y\cdot\left(\frac{x}{3}+\frac{y}{5}-1+\frac{x}{3}\right) = y\left(\frac{2}{3}x+\frac{y}{5}-\right.$

$f_y = x\left(\frac{x}{3}+\frac{y}{5}-1\right)+xy\cdot\frac{1}{5} = x\left(\frac{x}{3}+\frac{y}{5}-1+\frac{y}{5}\right) = x\left(\frac{2}{5}y+\frac{x}{3}-\right.$

$\nabla f = 0 \iff \begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=0 \\ \frac{y}{5}-1=0 \end{cases} \quad \begin{cases} y=0 \\ \frac{x}{3}-1=0 \end{cases} \quad \begin{cases} \frac{x}{3}+\frac{y}{5}-1+\frac{x}{3}=0 \\ \frac{x}{3}+\frac{y}{5}-1+\frac{y}{5}=0 \end{cases}$

$\iff (x,y) = (0,0),\ (0,5),\ (3,0),\ (1,5/3)$

$Hf(x,y) = \begin{pmatrix} 2/3\,y & \frac{2}{3}x+\frac{2}{5}y-1 \\ \frac{2}{3}x+\frac{2}{5}y-1 & 2/5\,x \end{pmatrix}$

$Hf(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ non def $\to$ sella

$Hf(0,5) = \begin{pmatrix} \cancel{4} & 1 \\ 1 & 0 \end{pmatrix}$ non def. $\to$ sella

$Hf(3,0) = \begin{pmatrix} 0 & 1 \\ 1 & \cancel{4} \end{pmatrix}$ non def. $\to$ sella

$Hf(1,5/3) = \begin{pmatrix} 10/9 & 2/3+2/3-1 \\ 1/3 & 2/5 \end{pmatrix}$: $\det = \frac{4}{9}-\frac{1}{9} > 0$

$10/9 > 0$

Def. pos. $\to$ p.to min rel.

(3) $\ddot{x} + 9\dot{x} = 1 + \sin(3t)$

$\ddot{z} + 9\dot{z} = 0 \qquad \lambda^2 + 9\lambda = 0 \qquad \lambda(\lambda+9) = 0 \qquad \lambda = 0,\ \lambda = -9$

$z(t) = A + B\cdot e^{-9t}$

$x_1(t) = Kt \Rightarrow \ddot{x}_1 + 9\dot{x}_1 = 9K = 1 \iff K = 1/9$

$x_2(t) = \cancel{K}\sin(3t) + D\cos(3t) \Rightarrow \ddot{x}_2 + 9\dot{x}_2 =$

$= -9C\sin(3t) - 9D\cos(3t) + 9\left[3C\cos(3t) - 3D\sin(3t)\right]$

$= \sin(3t)(-9C-27D) + \cos(3t)(-9D+27C) = \sin(3t)$

$\iff \begin{cases} -9D+27C=0 \\ -9C-27D=1 \end{cases} \iff \begin{cases} D=3C \\ -90\cdot C=1 \end{cases}$ | $x(t) = A + B\cdot e^{-9t} + \frac{1}{9}t - \frac{1}{\ldots}\sin(3t) - \frac{1}{\ldots}\cos(3t$

④ $g_{\phi}(r,\theta) = f(r\cos\theta, r\sin\theta)$; $f = f(x,y)$

$\partial_r g(r,\theta) = \partial_x f(r\cos\theta, r\sin\theta)\cdot\cos\theta + \partial_y f(r\cos\theta, r\sin\theta)\cdot\sin$

$\partial_{rr} g(r,\theta) = \partial_{xx} f(r\cos\theta, r\sin\theta)\cdot\cos^2\theta + \partial_{yx} f(r\cos\theta, r\sin\theta)\cdot\cos\theta\cdot\sin\theta$

$+ \partial_{xy} f(r\cos\theta, r\sin\theta)\cdot\sin\theta\cdot\cos\theta + \partial_{yy} f(r\cos\theta, r\sin\theta)\cdot\sin^2\theta$

$= \partial_{xx} f(r\cos\theta, r\sin\theta)\cdot\cos^2\theta + 2\cdot\partial_{xy} f(r\cos\theta, r\sin\theta)\cdot\sin\theta\cdot\cos\theta$

$+ \partial_{yy} f(r\cos\theta, r\sin\theta)\cdot\sin^2\theta.$

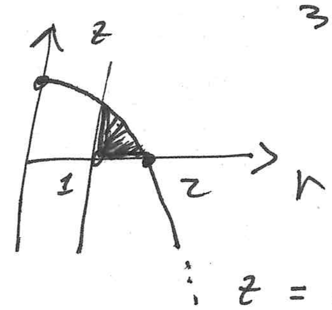
$\partial_r g(1,\pi/2) = \partial_y f(0,1)$

$\partial_{rr} g(1,\pi/2) = \partial_{yy} f(0,1)$

⑤ In coordinate cilindriche $\frac{x}{3} = r\cos\theta$, $\frac{y}{5} = r\sin\theta$:

$1 \le r^2 \le 4 - z$:

$r \ge 0$; $|\theta| \le \pi$



$A = \{(x,y) : 1 \le \frac{x^2}{9} + \frac{y^2}{25} \le 4\}$ e $\forall (x,y) \in A$,

$\alpha(x,y) = 0$; $\beta(x,y) = 4 - \left(\frac{x^2}{9} + \frac{y^2}{25}\right)$

⑧ Sia $w = z^2$: $w^2 - 9w + 81 = 0$ $\Delta = 81 - 4\cdot 81$
$= -3\cdot 81 < 0$

$w = \frac{9 \pm 9i\sqrt{3}}{2} = 9\cdot\frac{1 \pm i\sqrt{3}}{2}$

$z^2 = 9\cdot\frac{1+i\sqrt{3}}{2} = 9\cdot\left[\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]^{1/2} e^{i\alpha}$; $\alpha = \operatorname{arctg}(\sqrt{3}) =$

$\pi/3$. $= 9\cdot e^{i\pi/3}$

$z = \pm 3\cdot e^{i\pi/6} = \pm 3\cdot\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \pm 3\cdot\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

Similmente:

$$z^2 = 9 \cdot \frac{1 - i\sqrt{3}}{2} = 9 \cdot e^{i\alpha} \; ; \; \alpha = -\pi/3$$

$$z = \pm 3 \cdot e^{-i\pi/6} = \pm 3 \left(\frac{\sqrt{3}}{2} - i \frac{1}{2} \right)$$

cioè, $z = \pm 3 \left(\frac{\sqrt{3}}{2} \pm i \frac{1}{2} \right)$ (4 soluzioni).

(7) $$n^{2\gamma} \cdot \frac{(1 + 1/n)^{\gamma} - 1}{n} = \frac{n^{2\gamma} \cdot \left[1 + \gamma \cdot \frac{1}{n} + o\left(\frac{1}{n}\right) - 1 \right]}{n}$$

$$\underset{n \to \infty}{\sim} n^{2\gamma} \cdot \gamma \cdot \frac{1}{n^2} = \frac{\gamma}{n^{2-2\gamma}} :$$

convergenza $2 - 2\gamma > 1$ sse $\boxed{0 \leq \gamma < 1/2}$