Prova scritta di Complementi di Analisi Matematica L-S
2 febbraio 2011

Nome.. **Cognome**.. **Matricola**................

Scrivete solo le soluzioni e, se volete, i passaggi principali. Scrivete sul e consegnate solo il foglio degli esercizi.

(1) **[6 pti]** Sia $f : [0,\pi] \to \mathbb{R}$, $f(x) = \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|$. Trovare $\{c_n : \ n \in \mathbb{N}, \ n \geq 1\}$ in $\mathbb{C}$ tali per cui

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) \text{ in } L^2([0,\pi]).$$

$$c_n = \frac{4}{\pi n^2} \cdot \sin(n\pi/2) = \frac{4}{\pi n^2} \cdot \begin{cases} 0 & \text{se } n \text{ è pari} \\ (-1)^m & \text{se } n = 2m+1 \text{ è dispari} \end{cases}$$

$$f(x) = \sum_{m=0}^{+\infty} \frac{4 \cdot (-1)^m}{\pi(2m+1)^2} \cdot \sin((2m+1)x)$$

(2) **[6 pti]** Risolvere il problema di Cauchy:

$$\begin{cases} \partial_{xx} u(x,t) + \partial_t u(x,t) - \partial_{tt} u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(0,t) = u(\pi,t) = 0 \text{ per } t \in [0,\pi]; \\ u(x,0) = \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right| \text{ per } x \in [0,\pi]; \\ \partial_x u(x,0) = \frac{1}{2}\left(\frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|\right) \text{ per } x \in [0,\pi]. \end{cases}$$

t

$$U(x,t) = \frac{4}{\pi n^2} \cdot e^{t/2} \cdot \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}\right) \cdot \cos\left(\frac{\sqrt{4n^2-1}}{2} t\right) \cdot \sin(nx)$$

(3) [6 pti] Sia $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ la soluzione di $\partial_{xx} v(x, t)$

$$\begin{cases} \partial_t u(x,t) = \quad \text{in } \mathbb{R} \times [0,\infty), \\ u(x,0) = \begin{cases} 1 \text{ se } & -1 \leq x \leq 1, \\ 0 \text{ altrimenti.} \end{cases} \end{cases}$$

Sia $f : \mathbb{R} \to \mathbb{R}$ definita da $f(x) = u(x, 1)$. Calcolare $\hat{f}(\zeta) := \int_{-\infty}^{+\infty} f(x) e^{-i\zeta x} dx$.

$$\hat{f}(\zeta) = 2 \cdot \frac{\sin(\zeta)}{\zeta} \cdot e^{-\zeta^2}$$

(4) [6 pti] Risolvere il problema di Cauchy

$$\begin{cases} x\partial_x u(x,t) + t\partial_t u(x,t) = 0 \text{ in } \mathbb{R}^2; \\ u(x, \sqrt{1-x^2}) = \frac{\sqrt{1-x^2}}{x} \text{ per } 0 < x < 1 \in \mathbb{R}. \end{cases}$$

(Qual'è il dominio naturale della soluzione?)

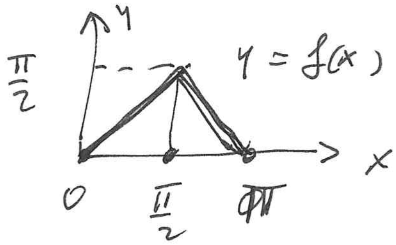
$$v(x,t) = \frac{t}{x} \qquad v : (0,1) \times \mathbb{R} \to \mathbb{R}$$

(5) [6 pti] Trovare tutte le funzioni u tali che

$$\begin{cases} \partial_{xx} u(x,t) + \partial_t u(x,t) - \partial_{tt} u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(x,0) = u(x,\pi) = 0 \text{ per } x \in [0,\pi]. \end{cases}$$

$$v(x,t) = e^{t} \cdot \sum_{n=1}^{\infty} \left[A_n \cdot \cos\left(\left(n^2 + \frac{1}{4}\right) x\right) + B_n \cdot \sin\left(\left(n^2 + \frac{1}{4}\right) x\right)\right] \cdot \sin(nt)$$

(1)



$$c_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin(nx)\,dx =$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cdot\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\sin(nx)\,dx\right\}$$

$$= \frac{2}{\pi n}\cdot\left\{\left[-\cos(nx)\cdot x\right]_0^{\pi/2} + \int_0^{\pi/2}\cos(nx)\,dx + \left[-\cos(nx)\cdot(\pi - x)\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\cos(nx)\,dx\right\}$$

$$= \frac{2}{\pi n}\cdot\left\{-\underline{\frac{\pi}{2}\cdot\cos\left(n\frac{\pi}{2}\right)} + \underline{\frac{\pi}{2}\cdot\cos\left(n\frac{\pi}{2}\right)} + \left[\frac{\sin(nx)}{n}\right]_0^{\pi/2} - \left[\frac{\sin(nx)}{n}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi n^2}\cdot 2\cdot\sin(n\pi/2) = \boxed{\frac{4}{\pi n^2}\cdot\sin(n\pi/2) = c_n}$$

(2) Cerco $v(x,t) = \sum_{n=1}^{\infty}\gamma_n(t)\cdot\sin(nx)$; per cui $v(0,t) = v(\pi,t) \equiv 0$.

$$0 = v_{xx} + v_t - v_{tt} = \sum_{n=1}^{\infty}\left[-n^2\gamma_n(t) + \dot\gamma_n(t) - \ddot\gamma_n(t)\right]\cdot\sin(nx)$$

$$\Rightarrow \quad \ddot\gamma_n - \dot\gamma_n + n^2\gamma_n = 0 \qquad \lambda^2 - \lambda + n^2 = 0$$

$$v(x,t) = \sum_{n=1}^{\infty} e^{\frac{t}{2}}\cdot\left[A_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + B_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]\cdot\sin(nx).$$

$$\lambda = \frac{1 \pm \sqrt{1 - 4n^2}}{2} = \frac{1}{2} \pm i\,\frac{\sqrt{4n^2-1}}{2} \quad \left(\begin{matrix} n \in \mathbb{N} \\ n \geq 1\end{matrix}\right)$$

$$\partial_t v(x,t) = \sum_{n=1}^{\infty}\dot\gamma_n(t)\cdot\sin(nx) \quad \text{e} \quad \partial_t v(x,0) = \sum_{n=1}^{\infty}\dot\gamma_n(0)\sin(nx).$$

Vogliamo quindi $\gamma_n(0) = c_n$ e $\dot\gamma_n(0) = c_n/2$

Cioè, $c_n = \gamma_n(0) = A_n$ e $\frac{c_n}{2} = \dot\gamma_n(0) = \frac{A_n}{2} + \frac{\sqrt{4n^2-1}}{2}\cdot B_n$:

$B_n = 0$.

$$\gamma_n(t) = e^{t/2}\cdot c_n\cdot\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right)$$

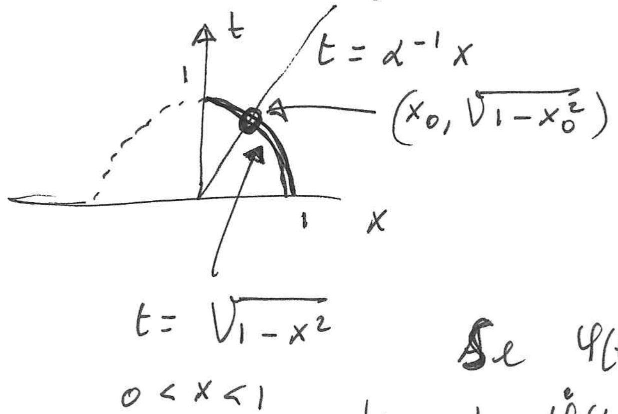
e

$$\boxed{v(x,t) = e^{t/2}\cdot\sum_{n=1}^{\infty}\frac{4}{\pi n^2}\cdot\sin\left(\frac{n\pi}{2}\right)\cdot\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right)\cdot\sin(nx)}$$

(3) Sia $g(x) = v(x,0)$: la soluzione dell'equazione del calore
$$\begin{cases} \partial_t v(x,t) = \partial_{xx} v(x,t) \\ v(x,0) = g(x) \end{cases}$$
verifica
$$\begin{cases} \partial_t \hat{v}(\xi,t) = -\xi^2 \cdot \hat{v}(\xi,t) \\ \hat{v}(\xi,0) = \hat{g}(\xi) \end{cases}$$
quindi (per la teoria vista o facendo i conti):
$$\hat{v}(\xi,t) = e^{-t\xi^2} \hat{g}(\xi).$$
Noi cerchiamo $\hat{f}(\xi) = \hat{v}(\xi,1) = e^{-\xi^2} \cdot \hat{g}(\xi)$.

Calcoliamo ora $\hat{g}(\xi) = \int_{-\infty}^{+\infty} v(x,0) \cdot e^{-i\xi x} dx = \int_{-1}^{1} e^{-i\xi x} dx$
$$= \left(\frac{e^{-i\xi x}}{-i\xi} \right)_{x=-1}^{x=1} = \frac{e^{i\xi} - e^{-i\xi}}{i\xi} = \frac{2}{\xi} \cdot \frac{e^{i\xi} - e^{-i\xi}}{2i} = \frac{2}{\xi} \cdot \sin(\xi)$$
Quindi: $\boxed{\hat{f}(\xi) = 2 \cdot \frac{\sin(\xi)}{\xi} \cdot e^{-\xi^2}}$

(4) Se $x = x(t)$ e $\varphi(t) = v(x(t),t)$, allora
$$\dot{\varphi}(t) = v_x \cdot \dot{x} + v_t = \frac{x}{t} v_x + v_t \iff \dot{x} = \frac{x}{t}$$
Risolvo: $\dot{x} = \frac{x}{t} \iff \frac{dx}{x} = \frac{dt}{t} \iff \log|x| = \log|t| + c \iff \boxed{x(t) = \alpha t}$



Curve caratteristiche: $x(t) = \alpha \cdot t$ $(\alpha > 0)$.

~~Passa~~ Passa per $(x_0, \sqrt{1-x_0^2})$ se
$$\alpha = \frac{x_0}{\sqrt{1-x_0^2}}.$$
Se $\varphi(t) = v(x(t),t)$ (con $x(\sqrt{1-x_0^2}) = x_0$), ho che $\dot{\varphi}(t) = \frac{1}{t} \cdot (x v_x + t v_t) = 0$ per l'equazione.

Quindi $\varphi(t) = K(x_0)$, costante rispetto a t.

Allora, $K(x_0) = \varphi(t) = v(x(t),t) = v(\alpha t, t) = v(x_0, \sqrt{1-x_0^2}) =$

$= \dfrac{\sqrt{1-x_0^2}}{x_0}$, cioè:

$v(x,t) = \dfrac{\sqrt{1-x_0^2}}{x_0} = \dfrac{1}{\alpha} = \dfrac{t}{x}$;

$$\boxed{\begin{array}{l} v(x,t) = \dfrac{t}{x} \\ v: (0,1)\times\mathbb{R} \to \mathbb{R} \end{array}}$$

Verifico: $x v_x + t v_t = x\left(-\frac{t}{x^2}\right) + t\cdot\frac{1}{x} = 0$

(5) Le soluzioni a variabili separate $v(x,t) = \varphi(x)\cdot\psi(t)$:

$0 = \varphi''(x)\cdot\psi(t) + \varphi(x)[\psi'(t) - \psi''(t)]$

$\dfrac{\varphi''(x)}{\varphi(x)} = \dfrac{\psi''(t) - \psi'(t)}{\psi(t)}$: dipendono da variabili indipendenti, quindi:

$\dfrac{\varphi''(x)}{\varphi(x)} = K = \dfrac{\psi''(t) - \psi'(t)}{\psi(t)}$; $\quad \psi(0) = \psi(\pi) = 0.$

$\begin{cases} \psi''(t) - \psi'(t) - K\cdot\psi(t) = 0 \\ \psi(0) = \psi(\pi) = 0 \end{cases}$

$\lambda^2 - \lambda - K = 0 \quad \Delta = 1 + 4K < 0$ voglio

$1 + 4K = -4n^2 \ (n \in \mathbb{N},\ n \geq 1)$

$\lambda = \dfrac{1 \pm 2in}{2}$

$K = \cancel{\dots} \ -n^2 - \frac{1}{4}$

$\Delta = -4n^2$

$\psi_n(t) = e^{\frac{t}{2}}\cdot \sin(nt)$.

$0 = \varphi_n''(x) - K\varphi_n(x) = \varphi_n''(x) + \left(n^2 + \frac{1}{4}\right)\varphi_n(x)$:

$$\boxed{v(x,t) = e^{t/2}\sum_{n=1}^{+\infty}\left[A_n\cdot\cos\left(\left(n^2+\tfrac{1}{4}\right)x\right) + B_n\cdot\sin\left(\left(n^2+\tfrac{1}{4}\right)x\right)\right]\cdot\sin(nt)}$$