Prova scritta di Analisi Matematica I
19 gennaio 2011

Nome..**Cognome**... **Matricola**................

Scrivete solo le soluzioni e, se volete, i passaggi principali. Scrivete sul e consegnate solo il foglio degli esercizi.

(1) **[6 pti]** Sia $f : [0, \pi] \to \mathbb{R}$, $f(x) = x \cdot (1 - x)$. Trovare $\{c_n : n \in \mathbb{N}, n \geq 1\}$ in $\mathbb{C}$ tali per cui

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) \text{ in } L^2([0, \pi]).$$

(2) **[6 pti]** Risolvere il problema di Cauchy:

$$\begin{cases} \partial_{xx}u(x,t) - \partial_t u(x,t) - \partial_{tt}u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(0,t) = u(\pi,t) = 0 \text{ per } t \in [0,\pi]; \\ u(x,0) = x \cdot (1-x) \text{ per } x \in [0,\pi]; \\ \partial_{\underset{t}{\cancel{x}}}u(x,0) = \frac{x\cdot(x-1)}{2} \text{ per } x \in [0,\pi]. \end{cases}$$

(3) **[6 pti]** Calcolare $f(\zeta) := \int_{-\infty}^{+\infty} f(x)e^{-i\zeta x}dx$ per $f(x) = x \cdot e^{-|x|}$.

(4) **[6 pti]** Sia $\varphi : \mathbb{R} \to \mathbb{R}$ una funzione di classe C^1 su $\mathbb{R}$. Risolvere il problema di Cauchy

$$\begin{cases} \partial_x u(x,t) + \partial_t u(x,t) = 0 \text{ per } (x,t) \in \mathbb{R} \times [0,\infty); \\ u(x,0) = \varphi(x) \text{ per } x \in \mathbb{R}. \end{cases}$$

(5) **[6 pti]** Trovare tutte le funzioni u tali che

$$\begin{cases} \partial_{xx} u(x,t) - \partial_t u(x,t) - \partial_{tt} u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(x,0) = u(x,\pi) = 0 \text{ per } x \in [0,\pi]. \end{cases}$$

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(1) $\frac{\pi}{2}\cdot c_n = \int_0^{\pi} \sin(nx)\cdot(x-x^2)\,dx =$

$= \left[-\frac{\cos(nx)}{n}\cdot(x-x^2)\right]_0^{\pi} + \int_0^{\pi}\frac{\cos(nx)}{n}\cdot(1-2x)\,dx$

$= \frac{\pi^2-\pi}{n}\cdot(-1)^n + \left[\frac{\sin(nx)}{n^2}\cdot(1-2x)\right]_0^{\pi} + \int_0^{\pi}\frac{\sin(nx)}{n^2}\cdot 2\,dx$

$= \frac{\pi^2-\pi}{n}\cdot(-1)^n - 2\left[\frac{\cos(nx)}{n^3}\right]_0^{\pi} = \frac{\pi^2-\pi}{n}\cdot(-1)^n - \frac{2}{n^3}\left[(-1)^n-1\right]$

$$\boxed{c_n = \frac{2}{\pi}\cdot\left\{\frac{\pi^2-\pi}{n}(-1)^n + \frac{2}{n^3}\left[1-(-1)^n\right]\right\}}$$

(2) Cerco $v(x,t) = \sum_{n=1}^{+\infty} c_n(t)\cdot\sin(nx)$

$0 = v_{xx} - v_t - v_{tt} = \sum_{n=1}^{+\infty}\left[-n^2c_n(t) - \dot c_n(t) - \ddot c_n(t)\right]\sin(nx)$ e

$x-x^2 = v(x,0) = \sum_{n=1}^{\infty} c_n(0)\cdot\sin(nx) \quad \forall x\in[0,\pi]$ e

$\frac{x-x^2}{2} = v_t(x,0) = \sum_{n=1}^{\infty}\dot c_n(0)\cdot\sin(nx) \quad \forall x\in[0,\pi]$

$$\Longleftrightarrow \begin{cases} \ddot c_n + \dot c_n + n^2 c_n = 0 & \forall n\geq 1;\ n\in\mathbb{N} \\ c_n(0) = c_n & \text{(come in (1))} \\ \dot c_n(0) = \frac{c_n}{2} \end{cases}$$

$\Delta = 1-4n^2<0$: $c_n(t) = \left[A_n\cdot\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + B_n\cdot\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]\cdot e^{-}$

$\lambda_n = -\frac{1}{2} \pm i\frac{\sqrt{4n^2-1}}{2}$ | $c_n(0) = A_n = c_n$ (quello di (1))

e $\dot c_n(0) = B_n\cdot\frac{\sqrt{4n^2-1}}{2} - \frac{A_n}{2} = \frac{c_n}{2}$ (quello di (1))

$A_n = c_n$ e $B_n = 0$

$$\boxed{v(x,t) = \sum_{n=1}^{\infty} c_n\cdot\left[\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + \right]\cdot e^{-t/2}\cdot\sin(nx)}$$

(3) $$\int_{-\infty}^{+\infty} x \cdot e^{-|x|} e^{-i\xi x}\, dx = \int_{0}^{+\infty} x \cdot e^{-x-i\xi x}\, dx + \int_{-\infty}^{0} x \cdot e^{x-i\xi x}\, dx$$

$$= \left(\frac{x \cdot e^{-x(1+i\xi)}}{-(1+i\xi)} \right)_{x=0}^{+\infty} \cancel{-} \int_0^{+\infty} \frac{e^{-x(1+i\xi)}}{-(1+i\xi)}\, dx + \left(\frac{x \cdot e^{x(1-i\xi)}}{1-i\xi} \right)_{-\infty}^{0}$$

$$- \int_{-\infty}^{0} \frac{e^{x(1-i\xi)}}{1-i\xi}\, dx = \cancel{-} \left[\frac{e^{-x(1+i\xi)}}{(1+i\xi)^2} \right]_0^{+\infty} - \left[\frac{e^{x(1-i\xi)}}{(1-i\xi)^2} \right]_{-\infty}^{0}$$

$$= + \frac{1}{(1+i\xi)^2} \cancel{-} \frac{1}{(1-i\xi)^2} = \frac{-(1+i\xi)^2 + (1-i\xi)^2}{(1-i\xi)^2 (1+i\xi)^2} =$$

$$= \frac{-4i\xi}{(1+\xi^2)^2} .$$

$$\boxed{\int_{-\infty}^{+\infty} x \cdot e^{-|x|} \cdot e^{-i\xi x}\, dx = \frac{-4i\xi}{(1+\xi^2)^2}}$$

(4) Pongo $\dot{x}(t) = 1$: $x(t) = t + x_0$

$h(t) \overset{def}{=} v(t+x_0, t)$ soddisfa $h'(t) = \partial_x v(t+x_0, t) + \partial_t v(t+x_0, t) = 0$

e $h(0) = v(x_0, 0) = \varphi(x_0)$

Quindi $h(t) = \varphi(x_0)$ $\forall t \in \mathbb{R}$:

$v(t+x_0, t) = \varphi(x_0)$ e poiché $t + x_0 = x$:

$$\boxed{v(x,t) = \varphi(x-t);\ (x,t) \in \mathbb{R} \times [0, +\infty)}$$

è la soluzione cercata.

(5) ~~Pongo~~ Separazione di variabili.

$v(x,t) = \varphi(x) \cdot \psi(t)$: $0 = \varphi''(x)\psi(t) - \varphi(x)(\psi'(t) + \psi''(t))$

$$\Longleftrightarrow \frac{\varphi''(x)}{\varphi(x)} = k = \frac{\psi''(t) + \psi'(t)}{\psi(t)} \qquad \exists k \in \mathbb{R}$$

Inoltre $0 = v(x,0) = \varphi(x) \cdot \psi(0) = v(x,\pi) = \varphi(x) \cdot \psi(\pi)$ $\forall x \in [0,\pi]$

cioè: $$\begin{cases} \psi''(t) + \psi'(t) - k \cdot \psi(t) = 0 \\ \psi(0) = \psi(\pi) = 0 \end{cases}$$

$\lambda^2 + \lambda - k = 0 \qquad \lambda = -\frac{1}{2} \pm \frac{\sqrt{1+4k}}{2}$

Voglio $1+4k = -4n^2$ $(n \in \mathbb{N},\ n \geq 1)$ così che

$\lambda = -\frac{1}{2} \pm in$ e $\psi(t) = e^{-\frac{t}{2}} \cdot \sin(nt)$.

Risolvo: $\frac{\varphi''(x)}{\varphi(x)} = -\frac{1+4n^2}{4} \qquad \varphi''(x) + \frac{1+4n^2}{4} \cdot \varphi(x) = 0$

$\lambda^2 + \frac{1+4n^2}{4} = 0$

$\cancel{\varphi(x) = A_n \cdot e^{\frac{\sqrt{1+4n^2}}{2}t} + B_n \cdot e^{-\frac{\sqrt{1+4n^2}}{2}t}}$

$\cancel{v(x,t) = \sum_{n=1}^{+\infty} A_n \cdot e^{\frac{\sqrt{1+4n^2}}{2}t} + B_n \cdot e^{-\frac{\sqrt{1+4n^2}}{2}t}}$

$\varphi(x) = A_n \cdot \cos\left(\frac{\sqrt{1+4n^2}}{4}\cancel{x}\right) + B_n \cdot \sin\left(\frac{\sqrt{1+4n^2}}{2}\cancel{x}\right)$

$$\boxed{\begin{array}{l} v(x,t) = \sum_{n=1}^{+\infty}\left[A_n \cdot \cos\left(\frac{\sqrt{1+4n^2}}{2}x\right) + B_n \cdot \sin\left(\frac{\sqrt{1+4n^2}}{2}x\right)\right] \cdot \sin(nt)\, e^{-\frac{t}{2}} \\ \qquad v : [0,\pi] \times [0,\pi] \to \mathbb{R} \\ \text{è la soluzione generale; } \{A_n\} \text{ e } \{B_n\} \text{ liberi.} \end{array}}$$