Prova scritta di Complementi di Analisi Matematica L-S
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Nome.. **Cognome**.. **Matricola**................

Scrivete solo le soluzioni e, se volete, i passaggi principali. Scrivete sul e consegnate solo il foglio degli esercizi.

(1) [**6 pti**] Sia $f : [0, \pi] \to \mathbb{R}$, $f(x) = \pi x - x^2$. Trovare $\{c_n : n \in \mathbb{N}, n \geq 1\}$ in $\mathbb{C}$ tali per cui

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) \text{ in } L^2([0, \pi]).$$

(2) [**6 pti**] Risolvere il problema di Cauchy:

$$\begin{cases} \partial_{xx} u(x,t) + \partial_t u(x,t) - \partial_{tt} u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(0,t) = u(\pi,t) = 0 \text{ per } t \in [0,\pi]; \\ u(x,0) = \pi x - x^2 \text{ per } x \in [0,\pi]; \\ \partial_t u(x,0) = \frac{1}{2}\left(\pi x - x^2\right) \text{ per } x \in [0,\pi]. \end{cases}$$

(3) [6 pti]. Sia $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ la soluzione di

$$(*) \begin{cases} \partial_{tt}u(x,t) - \partial_t u(x,t) = \partial_{xx}u(x,t) \text{ in } \mathbb{R} \times [0,\infty), \\ u(x,0) = \varphi(x), \\ \partial_t u(x,0) = 0. \end{cases}$$

Sia $u = u(x,t)$ la soluzione del problema $(*)$. Scrivere l'espressione per $v(\zeta, t) := \hat{u}(\zeta, t) = \int_{-\infty}^{\infty} u(x,t) e^{-i\zeta x} dx$.

(4) [6 pti] Risolvere il problema di Cauchy

$$\begin{cases} x\partial_t u(x,t) - t\partial_x u(x,t) = xu(x,t) \text{ in } \mathbb{R}^2; \\ u(x,0) = \sin(x) \text{ per } x \in \mathbb{R}. \end{cases}$$

(Qual'è il dominio naturale della soluzione?)

(5) [6 pti] Trovare tutte le funzioni u tali che

$$\begin{cases} \partial_{xx}u(x,t) + \partial_t u(x,t) - \partial_{tt}u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(x,0) = u(x,\pi) = 0 \text{ per } x \in [0,\pi]. \end{cases}$$

(1) $f(x) = \pi x - x^2, \quad x \in [0,\pi]$.

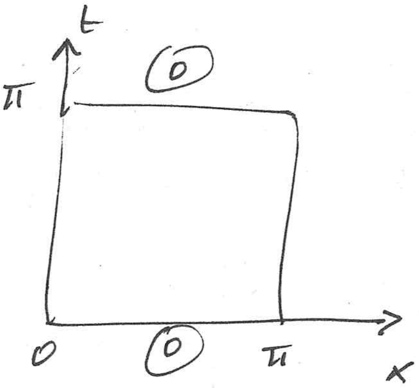
$$c_n = \frac{2}{\pi}\int_0^\pi (\pi x - x^2)\sin(nx)\,dx = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\cdot(\pi x - x^2)\right]_0^\pi + \frac{2}{\pi n}\int_0^\pi \cos(nx)\cdot(\pi - 2x)\,dx$$

$$= \frac{2}{\pi n}\cdot\left[\frac{\sin(nx)}{n}\cdot(\pi - 2x)\right]_0^\pi - \frac{2}{\pi n^2}\int_0^\pi (-2)\cdot\sin(nx)\,dx$$

$$= \frac{4}{\pi n^2}\cdot\left[-\frac{\cos(nx)}{n}\right]_0^\pi = \frac{4}{\pi n^3}\cdot[1-\cos(n\pi)] = \frac{4}{\pi n^3}[1-(-1)^n]$$

$$= \begin{cases} 8/\pi n^3 & \text{se } n \text{ è dispari} \\ 0 & \text{se } n \text{ è pari} \end{cases}$$

(5) ~~Provo con $v(x,t) = \sum_{n=1}^{\infty} c_n(t)\cdot\sin(nx)$~~,
~~che soddisfa $v(x,0) = v\ x, \pi$~~



Pongo $v(x,t) = \varphi(x)\psi(t)$
L'equazione diventa

$$0 = v_{xx} + v_t - v_{tt} = \varphi''(x)\psi(t) + \varphi(x)\psi'(t) - \varphi(x)\psi''(t), \text{ cioè}$$

$$\frac{\varphi''(x)}{\varphi(x)} = \frac{\psi''(t) - \psi'(t)}{\psi(t)}: \quad \begin{cases} \varphi''(x) = k\,\varphi(x) \\ \psi''(t) - \psi'(t) - k\,\psi(t) = 0 \\ \psi(0) = \psi(\pi) = 0 \end{cases}$$

$\lambda^2 - \lambda - k = 0$

$\Delta = 1 + 4k < 0: \quad \lambda = \frac{1}{2} \pm \frac{ih}{2}$

$1 + 4k = -h^2 \ (h \in \mathbb{R}) \quad \psi(t) = \left[A(h)\cdot\cos\left(\frac{h}{2}t\right) + B(h)\cdot\sin\left(\frac{h}{2}t\right)\right]\cdot e^{\frac{t}{2}}$

$0 = \psi(0) = A(h)$ e $0 = \psi(\pi) = B(h)\cdot e^{\pi/2}\cdot\sin\left(\frac{h}{2}\pi\right)$,

cioè, $h = 2n$, $n \in \mathbb{N}$, $n \geq 1$, e

$\psi_n(t) = B_n\cdot\sin(nt)\cdot e^{t/2}$ (posso porre $B_n = 1$).

$$K = \frac{-1-h^2}{4} = \frac{-1-4n^2}{4} = -\left(\frac{1}{4} + n^2\right)$$

$$\varphi''(x) + \left(n^2 + \frac{1}{4}\right)\varphi(x) = 0 \iff \varphi(x) = C_n\cos\left(\sqrt{n^2+\tfrac{1}{4}}\,x\right) + D_n\sin\left(\sqrt{n^2+\tfrac{1}{4}}\,x\right)$$

$$v(x,t) = e^{\frac{t}{2}}\sum_{n=1}^{\infty}\left[C_n\cdot\cos\left(\sqrt{n^2+\tfrac{1}{4}}\,x\right) + D_n\cdot\sin\left(\sqrt{n^2+\tfrac{1}{4}}\cdot x\right)\right]\cdot\sin(nt) \qquad (L.S) \qquad 2$$

con $C_n, D_n \in \mathbb{R}$ è la soluz. (generale) di (5).

(2) Provo con $v(x,t) = \sum_{n=1}^{\infty} c_n(t)\cdot\sin(nx)$,

così che avrò $v(0,t) = v(\pi,t) = 0 \quad \forall t \in [0,\pi]$.

L'equazione diventa:

$$0 = \sum_{n=1}^{\infty}\left(-n^2 c_n(t) + c_n'(t) - c_n''(t)\right)\cdot\sin(nx) = 0 :$$

$$c_n''(t) - c_n'(t) + n^2 c_n(t) = 0$$

La condizione $\pi x - x^2 = v(x,0) = \sum_{n=1}^{\infty} c_n(0)\cdot\sin(nx)$

diventa, per l'ES (1), $\quad c_n(0) = \frac{4}{\pi n^3}\cdot[1-(-1)^n]$

Ho poi

$$\underset{\frac{1}{2}(\pi x - x^2)}{\underset{\parallel}{\partial_t v(x,0)}} = \sum_{n=1}^{\infty} c_n'(0)\cdot\sin(nx), \quad \text{cioè}$$

$$c_n'(0) = \frac{2}{\pi n^3}[1-(-1)^n],$$

sempre per ES. (1).

$$\begin{cases} c_n''(t) - c_n'(t) + n^2\cdot c_n(t) = 0 \\ c_n(0) = \frac{4}{\pi n^3}[1-(-1)^n] \\ c_n'(0) = \frac{2}{\pi n^3}[1-(-1)^n] \end{cases}$$

$\lambda^2 - \lambda + n^2 = 0$

$\Delta = 1 - n^2$

$n=1$: $c_1(t) = A_1 e^{t/2} + B_1 t e^{t/2}$

$n \in \mathbb{N}$, $n > 1$:

$$c_n(t) = \left[A_n\cdot\cos\left(\frac{\sqrt{n^2-1}}{2}t\right) + B_n\cdot\sin\left(\frac{\sqrt{n^2-1}}{2}t\right)\right]\cdot e^{\frac{t}{2}}$$

$\frac{4}{\pi n^3}[1-(-1)^n]$

con $\cancel{0} = c_n(0) = A_n$ (se $n>1$) e $\quad 0 = c_1(0) = A_1$: $A_n = 0$ $\forall n \in$

Ho anche $\cancel{0} = c_n'(0) = \frac{1}{2}A_n + \frac{\sqrt{n^2-1}}{2}\cdot B_n$ (se $n>1$)

$\frac{2}{\pi n^3}[1-(-1)^n]$

e $0 = c_1'(0) = \frac{1}{2}A_1 + B_1$

cioè: $\begin{cases} A_n = \dfrac{4}{\pi n^3}[1-(-1)^n] \\ \dfrac{1}{2}A_n + \dfrac{\sqrt{n^2-1}}{2} B_n = \dfrac{4}{\pi n^3}[1-(-1)^n] \end{cases}$ se $n > 1$

$\begin{cases} A_1 = 0 \\ \frac{1}{2}A_1 + B_1 = 0 \end{cases}$

Da cui $A_n = \dfrac{4}{\pi n^3}[1-(-1)^n]$ e $B_n = 0$, se $n > 1$,

$A_1 = B_1 = 0$:

$$\boxed{v(x,t) = \sum_{n>1} \frac{4}{\pi n^3}[1-(-1)^n] \cdot \cos\left(\frac{\sqrt{n^2-1}}{2} t\right) \cdot \sin(nt) \cdot e^{t/2}}$$

(3) Ho che $(\partial_x v)^\wedge(\xi,t) =: \int_{-\infty}^{+\infty} \partial_x v(x,t)\, e^{-i\xi x}\, dx$

$= \int_{-\infty}^{+\infty} i\xi \cdot \partial_x v(x,t)\, e^{-i\xi x}\, dx$ (integrale per parti)

$= i\xi \cdot \hat{v}(\xi,t)$ e, analogamente,

$(\partial_{xx} v)^\wedge(\xi,t) = (i\xi)^2 \hat{v}(\xi,t) = -\xi^2 \hat{v}(\xi,t)$,

quindi l'equazione diventa (derivando rispetto a t sotto segno d'integrale):

$$\partial_{tt}\hat{v}(\xi,t) - \partial_t \hat{v}(\xi,t) = -\xi^2 \hat{v}(\xi,t)$$

Sia $y(t) = \hat{v}(\xi,t)$. Allora, $y'' - y' + \xi^2 y = 0$, con $\xi \in \mathbb{R}$ costante.

$\lambda^2 - \lambda + \xi^2 = 0$

$\Delta = 1 - 4\xi^2 > 0 \iff -\frac{1}{2} < \xi < \frac{1}{2}$

$$\lambda = \begin{cases} \dfrac{1 \pm \sqrt{1-4\xi^2}}{2} & \text{se } -1/2 < \xi < 1/2 \\ \dfrac{1}{2} \pm i\,\dfrac{\sqrt{4\xi^2-1}}{2} & \text{se } \xi < -1/2 \text{ o } \xi > 1/2 \end{cases}$$

Posso escludere il caso $\xi = -\frac{1}{2}, \frac{1}{2}$, che non vengono percepiti dalla formula d'inversione per la trasformata di Fourier.

Ne segue che $\hat{v}(\xi, t) = \begin{cases} A(\xi)\, e^{\frac{1+\sqrt{1-4\xi^2}}{2}t} + B(\xi)\, e^{\frac{1-\sqrt{1-4\xi^2}}{2}t} \\ \left[C(\xi)\cos\left(\frac{\sqrt{4\xi^2-1}}{2}t\right) + D(\xi)\sin\left(\frac{\sqrt{4\xi^2-1}}{2}t\right)\right]\cdot e \end{cases}$

se $\begin{cases} |\xi| < 1/2 \\ |\xi| > 1/2 \end{cases}$ e

$$\hat{\varphi}(\xi) = \hat{v}(\xi, 0) = \begin{cases} A(\xi) + B(\xi) & \text{se } |\xi| < 1/2 \\ C(\xi) & \text{se } |\xi| > 1/2 \end{cases}$$

$$\text{e } 0 = \partial_t \hat{v}(\xi, 0) = \begin{cases} \frac{1+\sqrt{1-4\xi^2}}{2}\cdot A(\xi) + \frac{1-\sqrt{1-4\xi^2}}{2}\cdot B(\xi) & \text{se } |\xi| < 1/2 \\ \frac{1}{2}C(\xi) + \frac{\sqrt{4\xi^2-1}}{2}\cdot D(\xi) & \text{se } |\xi| > 1/2 \end{cases}$$

Per $|\xi| < 1/2$: $\begin{cases} A(\xi) + B(\xi) = \hat{\varphi}(\xi) \\ \frac{A(\xi) + B(\xi)}{2} + \frac{\sqrt{1-4\xi^2}}{2}\left(A(\xi) - B(\xi)\right) = 0 \end{cases}$

$$\begin{cases} A(\xi) + B(\xi) = \hat{\varphi}(\xi) \\ A(\xi) - B(\xi) = -\frac{\hat{\varphi}(\xi)}{2}\cdot\frac{2}{\sqrt{1-4\xi^2}} \end{cases} \qquad \begin{cases} A(\xi) = \frac{\hat{\varphi}(\xi)}{2}\cdot\left(1 - \frac{1}{\sqrt{1-4\xi^2}}\right) \\ B(\xi) = \frac{\hat{\varphi}(\xi)}{2}\left(1 + \frac{1}{\sqrt{1-4\xi^2}}\right) \end{cases}$$

Per $|\xi| > \frac{1}{2}$: $\begin{cases} C(\xi) = \hat{\varphi}(\xi) \\ \frac{1}{2}C(\xi) + \frac{\sqrt{4\xi^2-1}}{2}D(\xi) = 0 \end{cases}$

$$C(\xi) = \hat{\varphi}(\xi)$$

$$D(\xi) = -\frac{\hat{\varphi}(\xi)}{2}\cdot\frac{2}{\sqrt{4\xi^2-1}} = -\frac{\hat{\varphi}(\xi)}{\sqrt{4\xi^2-1}}$$

$$\Rightarrow \hat{v}(\xi, t) = \begin{cases} \frac{\hat{\varphi}(\xi)}{2}\left[\left(1 - \frac{1}{\sqrt{1-4\xi^2}}\right)\cdot e^{\frac{1+\sqrt{1-4\xi^2}}{2}t} + \left(1 + \frac{1}{\sqrt{1-4\xi^2}}\right)\cdot e^{\frac{1-\sqrt{1-4\xi^2}}{2}t}\right] & \text{se } |\xi| < 1/2 \\ e^{t/2}\cdot\hat{\varphi}(\xi)\cdot\left[\cos\left(\frac{\sqrt{4\xi^2-1}}{2}t\right) - \frac{1}{\sqrt{4\xi^2-1}}\sin\left(\frac{\sqrt{4\xi^2-1}}{2}t\right)\right] & \text{se } |\xi| > \frac{1}{2} \end{cases}$$

4) $\partial_t v(x,t) - \frac{t}{x} \cdot \partial_x v(x,t) = v(x,t)$

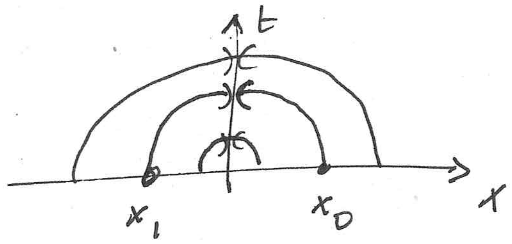
$\frac{d}{dt} v(x(t),t) = v_x \cdot \dot{x} + v_t$; equazione curve caratteristiche: $\begin{cases} \dot{x} = -t/x \\ x(0) = x_0 \end{cases}$

$x\dot{x} + t = 0 \qquad \frac{d}{dt}\left(\frac{x^2+t^2}{2}\right) = 0$

$x^2+t^2 = c$, ma $x_0^2+0^2 = c$, cioè: $x^2+t^2 = x_0^2$

$x = +\sqrt{x_0^2-t^2}$ (se $x_0 > 0$)

$x = -\sqrt{x_0^2-t^2}$ (se $x_0 < 0$)



Sia $\varphi(t) = v(x(t),t)$ con $x(0) = x_0$.

Allora, $\varphi(0) = v(x(0),0) = v(x_0,0) = \sin(x_0)$

$\begin{cases} \dot{\varphi}(t) = \varphi(t) \\ \varphi(0) = \sin(x_0) \end{cases}$ dall'equazione

$\varphi(t) = c \cdot e^t$ e $\sin(x_0) = \varphi(0) = c \cdot e^0 = c$

Ho $\varphi(t) = \sin(x_0) \cdot e^t$

Avendo $x^2+t^2 = x_0^2$, $x_0 = \begin{cases} \sqrt{x^2+t^2} & (\text{se } x>0) \\ -\sqrt{x^2+t^2} & (\text{se } x<0) \end{cases}$

Quindi $\boxed{v(x,t) = \sin(x_0) \cdot e^t = \begin{cases} \sin\left(\sqrt{x^2+t^2}\right) \cdot e^t & (x>0) \\ -\sin\left(\sqrt{x^2+t^2}\right) \cdot e^t & (x<0) \end{cases}}$