Prova scritta di Complementi di Analisi Matematica L-S
23 gennaio 2012

Nome...Cognome... Matricola.................

Scrivete solo le soluzioni e, se volete, i passaggi principali. Scrivete sul e consegnate solo il foglio degli esercizi.

(1) [**6 pti**] Sia $f : [0, \pi] \to \mathbb{R}$,

$$f(x) = \begin{cases} 0 \text{ se } 0 \leq x < \frac{\pi}{2} \\ 1 \text{ se } \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

Trovare $\{k_n : n \in \mathbb{N},\ n \geq 1\}$ in $\mathbb{C}$ tali per cui

$$f(x) = \sum_{n=1}^{\infty} k_n \sin(nx) \text{ in } L^2([0, \pi]).$$

(2) [**6 pti**] Risolvere il problema di Cauchy:

$$\begin{cases} \partial_{xx} u(x,t) + \partial_t u(x,t) - \partial_{tt} u(x,t) = 0 \text{ per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(0,t) = u(\pi,t) = 0 \text{ per } t \in [0,\pi]; \\ u(x,0) = \begin{cases} 0 \text{ se } 0 \leq x < \frac{\pi}{2} \\ 1 \text{ se } \frac{\pi}{2} \leq x \leq \pi \end{cases} ; \\ \partial_t u(x,0) = \begin{cases} 0 \text{ se } 0 \leq x < \frac{\pi}{2} \\ \frac{1}{2} \text{ se } \frac{\pi}{2} \leq x \leq \pi \end{cases} \end{cases}$$

(3) [6 pti] Sia $u : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ la soluzione di

$$\begin{cases} \partial_{tt} u(x,t) = \partial_{xx} u(x,t) \text{ in } \mathbb{R} \times [0,\infty), \\ u(x,0) = \begin{cases} 1 \text{ se } -1 \leq x \leq 1, \\ 0 \text{ altrimenti.} \end{cases} \\ u_t(x,0) = 0 \end{cases}$$

Sia $f : \mathbb{R} \to \mathbb{R}$ definita da $f(x) = u(x,1)$. Calcolare $\hat{f}(\zeta) := \int_{-\infty}^{+\infty} f(x) e^{-i\zeta x} dx$.

(4) [6 pti] Risolvere il problema di Cauchy

$$\begin{cases} (1+x^2)u_x + u_t = u + 1, \\ u(x,0) = e^{-x^2}. \end{cases}$$

(5) [6 pti] Trovare tutte le funzioni u tali che

$$\begin{cases} \partial_{xx} u(x,t) + \partial_t u(x,t) - \partial_{tt} u(x,t) = \begin{cases} 0 \text{ se } 0 \leq x < \frac{\pi}{2} \\ 1 \text{ se } \frac{\pi}{2} \leq x \leq \pi \end{cases} \quad \text{per } (x,t) \in [0,\pi] \times [0,\pi]; \\ u(0,t) = u(\pi,t) = 0 \text{ per } t \in [0,\pi]. \end{cases}$$

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(1) $k_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx =$

$= \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)\,dx = \frac{2}{\pi n}\left[-\cos(nx)\right]_{\pi/2}^{\pi}$

$= \boxed{\frac{2}{\pi n}\cdot\left[\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right]} = \begin{cases} \frac{+2\ (-1)^{2m}}{\pi(2m+1)} & \text{se } n = 2m+1,\ m \geq 0 \\ \frac{2}{\pi\cdot 2m}\left[\cos(m\pi) - 1\right] & \text{se } n = 2m,\ m \geq 1 \end{cases}$

ultimi punti di vista su k_n →

$= \begin{cases} \frac{2}{\pi(2m+1)} & \text{se } n = 2m+1;\ m \geq 0 \\ \frac{(-1)^m - 1}{\pi m} & \text{se } n = 2m;\ m \geq 1 \end{cases} = \begin{cases} \frac{2}{\pi(2m+1)} & \text{se } n = 2m+1,\ m \geq 0 \\ -\frac{2}{\pi \ell} & \text{se } n = 4\ell,\ \ell \geq 1 \\ 0 & \text{se } n = 4\ell+2,\ \ell \geq 0 \end{cases}$

(2) Provo con $v(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(nx)$, così che $v(0,t) = v(\pi,t)\ \forall t \in \mathbb{R}$.

$0 = v_{xx}(x,t) + v_t(x,t) - v_{tt}(x,t) = \sum_{n=1}^{\infty}\left[-n^2 c_n(t) + c_n'(t) - c_n''(t)\right]\cdot\sin(nx)$

$\Leftrightarrow \quad -n^2 c_n(t) + c_n'(t) - c_n''(t) = 0 \quad \forall n \in \mathbb{N}$

$v(x,0) = \sum_{n=1}^{\infty} c_n(0)\sin(nx) = \underbrace{f(x) = \sum_{n=1}^{\infty} k_n\cdot\sin(nx)}_{\text{come in (1)}}$

$\Leftrightarrow \quad c_n(0) = k_n \quad \forall n \in \mathbb{N}$

$v_t(x,0) = \sum_{n=1}^{\infty} c_n'(0)\sin(nx) = \frac{f(x)}{2} = \sum_{n=1}^{\infty}\frac{k_n}{2}\sin(nx)$

$\Leftrightarrow \quad c_n'(0) = \frac{k_n}{2} \quad \forall n \in \mathbb{N}$

$\begin{cases} c_n''(t) - c_n'(t) + n^2 c_n(t) = 0 \\ c_n(0) = k_n \\ c_n'(0) = \frac{k_n}{2} \end{cases} \qquad \begin{aligned} &\lambda^2 - \lambda + n^2 = 0 \\ &\Delta = 1 - 4n^2 < 0 \\ &\lambda = \tfrac{1}{2} \pm i\,\tfrac{\sqrt{4n^2-1}}{2} \end{aligned}$

$c_n(t) = e^{t/2}\left[A_n\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right) + B_n\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]$

$c_n'(t) = e^{t/2}\left[\left(\frac{1}{2}A_n + \frac{\sqrt{4n^2-1}}{2}B_n\right)\cos\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right.$
$\left. + \left(\frac{1}{2}B_n - \frac{\sqrt{4n^2-1}}{2}A_n\right)\sin\left(\frac{\sqrt{4n^2-1}}{2}t\right)\right]$

$$\begin{cases} k_n = c_n(0) = A_n \\ \dfrac{k_n}{2} = c_n'(0) = \dfrac{A_n}{2} + \dfrac{\sqrt{4n^2-1}}{2} B_n \end{cases} \qquad \begin{cases} A_n = k_n \\ B_n = 0 \end{cases}$$

$$c_n(t) = k_n\, e^{t/2} \cdot \cos\left(\frac{\sqrt{4n^2-1}}{2}\, t\right)$$

$$\boxed{v(x,t) = e^{\frac{t}{2}} \sum_{n=1}^{\infty} k_n \cos\left(\frac{\sqrt{4n^2-1}}{2}\, t\right) \cdot \sin(nx)}$$

(3) Sia $\hat{v}(\xi,t) = \int_{-\infty}^{+\infty} v(x,t)\, e^{-i\xi x}\, dx$, così che

$$\widehat{v_x}(\xi,t) = \int_{-\infty}^{+\infty} v_x(x,t)\, e^{-i\xi x}\, dx \overset{\text{I.P.}}{=} -\int_{-\infty}^{+\infty} v(x,t) \cdot \partial_x\left(e^{-i\xi x}\right) dx$$

$$= i\xi \int_{-\infty}^{+\infty} v(x,t)\, e^{-i\xi x}\, dx = i\xi\, \hat{v}(\xi,t)$$

e $\widehat{v_{xx}}(\xi,t) = i\xi\, \widehat{v_x}(\xi,t) = (i\xi)^2\, \hat{v}(\xi,t) = -\xi^2\, \hat{v}(\xi,t)$.

D'altra parte $\widehat{\partial_t v}(\xi,t) = \int_{-\infty}^{+\infty} \partial_t v(x,t)\, e^{-i\xi x}\, dx$

$= \partial_t \int_{-\infty}^{+\infty} v(x,t)\, e^{-i\xi x}\, dx$ (derivata sotto segno d'integrale)

$= \partial_t\, \hat{v}(\xi,t)$ e $\widehat{\partial_{tt} v}(\xi,t) = \partial_t\left[\widehat{\partial_t v}\right](\xi,t) = \partial_{tt}\hat{v}(\xi,t)$

L'equazione diventa $\boxed{\partial_{tt}\hat{v}(\xi,t) + \xi^2\, \hat{v}(\xi,t) = 0}$

Posto $\varphi(t) = \hat{v}(\xi,t)$; ciò significa $\ddot{\varphi}(t) + \xi^2 \varphi(t) = 0$

cioè, $\varphi(t) = A(\xi)\cos(\xi t) + B(\xi)\sin(\xi t)$

$$\boxed{\hat{v}(\xi,t) = A(\xi)\cos(\xi t) + B(\xi)\sin(\xi t)}$$

$$\partial_t \hat{v}(\xi,t) = -\xi \cdot A(\xi)\sin(\xi t) + \xi B(\xi)\cos(\xi t)$$

$\Rightarrow \partial_t\hat{v}(\xi,0) = \widehat{\partial_t v}(\xi,0) = \xi B(\xi) = 0$ per la seconda condizione iniziale

$\Rightarrow B(\xi) \equiv 0 \Rightarrow \boxed{\hat{v}(\xi,t) = A(\xi) \cdot \cos(\xi t)}$

La prima condizione iniziale implica che

$$A(\xi) = \hat{v}(\xi,0) = \int_{-1}^{1} e^{-i\xi x}\,dx = \left[\frac{e^{-i\xi x}}{-i\xi}\right]_{-1}^{1} =$$

$$= \frac{e^{-i\xi}-e^{i\xi}}{-i\xi} = \frac{e^{i\xi}-e^{-i\xi}}{2i}\cdot\frac{2}{\xi} = \frac{2\sin(\xi)}{\xi}$$

poichè $\left.\begin{aligned} e^{i\xi} &= \cos\xi + i\sin\xi \\ e^{-i\xi} &= \cos\xi - i\sin\xi \end{aligned}\right\} \Rightarrow e^{i\xi}-e^{-i\xi} = 2i\sin\xi$

Ho quindi che $\boxed{\hat{v}(\xi,t) = \frac{2\cdot\sin(\xi)}{\xi}\cdot\cos(\xi t)}$

Ne segue che $\hat{f}(\xi) = \hat{v}(\xi,1) = \frac{2\sin(\xi)\cos(\xi)}{\xi} = \frac{\sin(2\xi)}{\xi}$

$$\boxed{\hat{f}(\xi) = \frac{\sin(2\xi)}{\xi}}$$

(4) Se $x = x(t)$, $\frac{d}{dt}v(x(t),t) = v_x(x,t)\cdot\dot{x} + v_t(x,t)$

$\Rightarrow$ $\dot{x} = x^2+1$ è l'eq. delle curve caratteristiche.

Risolvo $\begin{cases}\dot{x} = x^2+1 \\ x(0) = x_0\end{cases}$ $\begin{cases}\frac{dx}{x^2+1} = dt \\ x(0) = x_0\end{cases}$

$$t = \int_0^t ds = \int_{x(0)}^{x(t)} \frac{dy}{y^2+1} = \operatorname{arctg}(x(t)) - \operatorname{arctg}(x_0)$$

(A) $\boxed{t = \operatorname{arctg}(x(t)) - \operatorname{arctg}(x_0)}$

Sia $\varphi(t) = v(x(t),t)$. Allora $\dot{\varphi}(t) = \varphi(t)+1$ (Eq. Diff.)

$\varphi(0) = v(x(0),0) = v(x_0,0) = e^{-x_0^2}$

$\begin{cases}\dot{\varphi}(t) = \varphi(t)+1 \\ \varphi(0) = e^{-x_0^2}\end{cases}$ $\begin{cases}\dot{\varphi}(t) - \varphi(t) = 1 \\ \varphi(0) = e^{-x_0^2}\end{cases}$ $\begin{cases}\frac{d}{dt}\left[e^{-t}\varphi(t)\right] = e^{-t}\left[\dot{\varphi}(t) - \varphi(t)\right] = e^{-t} \\ \text{e} \quad \varphi(0) = e^{-x_0^2}\end{cases}$

$$\left[e^{-s}\varphi(s)\right]_{s=0}^{s=t} \overset{(eq.)}{=} \int_0^t e^{-t}\,ds = \left[-e^{-s}\right]_0^t = 1 - e^{-t}$$

$$\| \quad e^{-t}\varphi(t) - \varphi(0) = e^{-t}\varphi(t) - e^{-x_0^2}$$

$$\Rightarrow v(x(t),t) = \varphi(t) = e^t[1-e^{-t}+e^{-x_0^2}] =$$
$$= e^t - 1 + e^t \cdot e^{-x_0^2}$$

Ma da (#), $\operatorname{arctg}(x_0) = \operatorname{arctg}(x(t)) - t$

$$\Rightarrow \boxed{x_0 = tg(\operatorname{arctg}(x(t)) - t)}$$

$$\Rightarrow \boxed{v(x,t) = e^t - 1 + e^t \cdot e^{-tg^2(\operatorname{arctg}(x)-t)}}$$

(5) Sia $f = f(x)$ come in (1); $f(x) = \sum_{n=1}^{\infty} k_n \sin(nx)$.

Posto $v(x,t) = \sum_{n=1}^{\infty} c_n(t) \sin(nx)$,

le condizioni $v(0,t) = v(\pi,t) = 0$ sono soddisfatte

e $v_{xx} + v_t - v_{tt} = f(x)$ diventa

(ragionando come in (2))

$$\sum_{n=1}^{\infty} [-n^2 c_n(t) + \dot{c}_n(t) - \ddot{c}_n(t)] \sin(nx) = \sum_{n=1}^{\infty} k_n \sin(nx)$$

Vogliamo risolvere dunque

$$\boxed{\ddot{c}_n(t) - \dot{c}_n(t) + n^2 c_n(t) = k_n}$$

Provo con $c_n(t) = c_n$ costante:

$$n^2 c_n = k_n \iff c_n = \frac{k_n}{n^2}$$

Aggiungendo le soluzioni dell'omogenea associata ($\ddot{z}_n - \dot{z}_n + n^2 z = 0$) trovate in (2):

$$c_n(t) = e^{t/2}\left[A_n \cos\left(\frac{\sqrt{n^2-1}}{2}t\right) + B_n \sin\left(\frac{\sqrt{n^2-1}}{2}t\right)\right] + \frac{k_n}{n^2}$$

Quindi:

$$\boxed{v(x,t) = \sum_{n=1}^{\infty}\left\{e^{t/2}\left[A_n \cos\left(\frac{\sqrt{n^2-1}}{2}t\right) + B_n \sin\left(\frac{\sqrt{n^2-1}}{2}t\right)\right] + \frac{k_n}{n^2}\right\}\sin(nx)}$$

con $A_n, B_n \in \mathbb{R}$ a scelta libera.