CAMLS 21/11/2011

(1) Risolvere
$$\begin{cases} u_{tt} - u_{xx} = 0 & (x,t) \in \mathbb{R}^2 \\ u(x,0) = \begin{cases} 1 & \text{se } |x| \le \pi \\ 0 & \text{se } |x| \ge \pi \end{cases} \\ u_t(x,0) = \begin{cases} \pi - |x| & \text{se } |x| \le \pi \\ 0 & \text{se } |x| \ge \pi \end{cases} \end{cases}$$

(2) Sia u la soluzione di (1) e sia
$f : \mathbb{R} \to \mathbb{R}$, $f(x) = u(x,2)$.
Calcolare $\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)\, e^{-ix\xi}\, dx$.

(3) Risolvere
$$u_t - u_{xx} - u_x = 0 \qquad (x,t) \in [0,\pi] \times \mathbb{R}^+$$
$$u(0,t) = u(\pi,t) = 0$$
$$u(x,0) = \begin{cases} 1 & \text{se } 0 \le x \le \pi/2 \\ -1 & \text{se } \pi/2 < x \le \pi \end{cases}$$

(4) Risolvere
$$\begin{cases} u_x(x,t) \cdot (1+x^2) + u_t(x,t) + u(x,t) \cdot t = 0 \\ u(x,0) = x \end{cases}$$

Svolgimento.

(1) Sia $\hat{v}(\xi,t) = \int_{-\infty}^{+\infty} v(x,t)\, e^{-ix\xi}\, dx$

$$\Rightarrow (\partial_x v)^\wedge(\xi,t) = \int_{-\infty}^{+\infty} \partial_x v(x,t) e^{-ix\xi} dx = -\int_{-\infty}^{+\infty} v(x,t)\, \partial_x(e^{-ix\xi})\, dx$$

(integrando per parti)

$$= i\xi \int_{-\infty}^{+\infty} v(x,t)\, e^{-ix\xi}\, dx = i\xi \cdot \hat{v}(\xi,t)$$

e $(\partial_{xx} v)^\wedge(\xi,t) = i\xi(\partial_x v)^\wedge(\xi,t) = (i\xi)^2 \hat{v}(\xi,t) = -\xi^2 \hat{v}(\xi,t)$

$$v_{tt} - v_{xx} = 0 \iff \partial_{tt}\hat{v}(\xi,t) + \xi^2 \hat{v}(\xi,t) = 0$$

$$\iff \hat{v}(\xi,t) = A(\xi)\cos(\xi t) + B(\xi)\sin(\xi t)$$

(Risolvo una equazione differenziale ordinaria in t)

Ho che $\hat{v}(\xi,0) = A(\xi)$ e $\partial_t \hat{v}(\xi,0) = \xi \cdot B(\xi)$,

dove $\hat{v}(\xi,0) = \int_{-\pi}^{\pi} e^{-i\xi x}\, dx = \left.\frac{e^{-i\xi x}}{-i\xi}\right|_{-\pi}^{\pi} = \frac{e^{-i\xi\pi} - e^{i\xi\pi}}{-i\xi}$

$$= \frac{2\pi}{\pi\xi} \cdot \frac{e^{i\xi\pi} - e^{-i\xi\pi}}{2i} = 2\pi \cdot \frac{\sin(\pi\xi)}{\pi\xi}.$$

e $\hat{v}_t(\xi,0) = \int_{-\pi}^{\pi} (\pi - |x|)\, e^{-i\xi x}\, dx = \int_{-\pi}^{\pi} (\pi - |x|)\,[\cos(\xi x) - i\sin(\xi x)]\, dx$

$$= \int_{-\pi}^{\pi} (\pi - |x|)\cdot \cos(\xi x)\, dx \quad (\text{perché } \pi - |x| \text{ è pari})$$

$$= 2\cdot\int_0^{\pi} (\pi - x)\cos(\xi x)\, dx = 2\cdot\left[(\pi - x)\cdot\frac{\sin(\xi x)}{\xi}\right]_0^{\pi} - 2\int_0^{\pi} \frac{\sin(\xi x)}{\xi}(-1)\, dx$$

$$= \frac{2}{\xi}\cdot\left[-\frac{\cos(\xi x)}{\xi}\right]_0^{\pi} = \frac{2}{\xi^2}[1 - \cos(\xi\pi)].$$

Quindi
$$\boxed{\begin{aligned}\hat{v}(\xi,t) &= \hat{v}(\xi,0)\cdot\cos(\xi t) + \xi^{-1}\partial_t\hat{v}(\xi,0)\cdot\sin(\xi t)\\ &= 2\pi\cdot\frac{\sin(\pi\xi)}{\pi\xi}\cdot\cos(\xi t) + \frac{2}{\xi^3}[1 - \cos(\xi\pi)]\cdot\sin(\xi t)\end{aligned}}$$

e $\boxed{v(x,t) = 2\pi\cdot\int_{-\infty}^{+\infty} \hat{v}(\xi,t)\, e^{ix\xi}\, d\xi.}$

(2) In particolare: $\hat{v}(\xi,2) = \frac{2\sin(\pi\xi)}{\xi}\cdot\cos(2\xi) + \frac{2}{\xi^3}[1 - \cos(\xi\pi)]\cdot\sin(2\xi)$

(3) $v(x,t) = \varphi(x)\psi(t)$: $\psi'(t)\varphi(x) = \psi(t)(\varphi''(x) + \varphi'(x))$

$$\frac{\psi'(t)}{\psi(t)} = \frac{\varphi''(x)+\varphi'(x)}{\varphi(x)} \Rightarrow \begin{cases} \varphi''(x)+\varphi'(x)+K\cdot\varphi(x)=0 \\ \varphi(0)=\varphi(\pi)=0 \end{cases} \text{ e } \left\{ \frac{\psi'(t)}{\psi(t)} = -K \right.$$

$\lambda^2+\lambda+K=0 \qquad \Delta = 1-4K = -4n^2 \quad (n\in\mathbb{N},\ n\geq 1)$

$K = \frac{1+4n^2}{4}$ e $\lambda = \frac{-1}{2} \pm i\, n$

$\varphi(x) = e^{-x/2}\cdot A_n \sin(nx)$

$\psi'(t) = -\frac{1+4n^2}{4}\cdot\psi(t) \qquad \psi(t) = e^{-\frac{1+4n^2}{4}t}$

$$v(x,t) = \sum_{n=1}^{\infty} e^{-x/2}\cdot e^{-\frac{1+4n^2}{4}t}\cdot A_n \sin(nx)$$

$$v(x,0) = e^{-x/2}\cdot\sum_{n=1}^{\infty} A_n\cdot\sin(nx) = \begin{cases} 1 & \text{su } [0,\pi/2] \\ -1 & \text{su } [\pi/2,\pi] \end{cases}$$

$$\Leftrightarrow \sum_{n=1}^{\infty} A_n\cdot\sin(nx) = f(x) = e^{+x/2}\cdot\begin{cases} 1 & \text{su } [0,\pi/2] \\ -1 & \text{su } [\pi/2,\pi) \end{cases}$$

$$\Leftrightarrow A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} e^{x/2}\cdot\sin(nx)\,dx + -\int_{\pi/2}^{\pi} e^{x/2}\cdot\sin(nx)\,dx\right].$$

Calcolo le primitive integrando per parti:

$$\int e^{x/2}\sin(nx)\,dx = 2\cdot e^{x/2}\sin(nx) - \int 2n\cdot e^{x/2}\cdot\cos(nx)\,dx$$
$$= 2\cdot e^{x/2}\sin(nx) - n\cdot\left[2e^{x/2}\cos(nx) + 2n\cdot\int e^{x/2}\sin(nx)\,dx\right]$$
$$= 2e^{x/2}\cdot[\sin(nx) - 2n\cdot\cos(nx)] - 4n\cdot\int e^{x/2}\sin(nx)\,dx$$
$$\Rightarrow (1+4n)\cdot\int e^{x/2}\cdot\sin(nx)\,dx = 2\cdot e^{x/2}[\sin(nx) - 2n\cdot\cos(nx)]$$
$$\Rightarrow \int e^{x/2}\cdot\sin(nx)\,dx = \frac{2}{1+4n}\cdot e^{x/2}[\sin(nx) - 2n\cdot\cos(nx)]$$
$$\stackrel{dF}{=} \frac{2\cdot F_n(x)}{1+4n}$$ (per non dover scrivere l'espressione più volte).

$$\Rightarrow A_n = \frac{2^2}{\pi(1+4n)}\cdot\left(F_n(\pi/2) - F_n(0)\right) - \left(F_n(\pi) - F_n(\pi/2)\right)$$
$$= \frac{2^2}{\pi\cdot(1+4n)}\cdot\left[2F_n(\pi/2) - F_n(0) - F_n(\pi)\right]$$

$$\boxed{v(x,t) = e^{-x/2}\cdot\sum_{n=1}^{\infty}\frac{2^2}{\pi(1+4n)}\cdot\left[2F_n(\pi/2) - F_n(0) - F_n(\pi)\right]\cdot e^{-\frac{1+4n^2}{4}t}\cdot\sin(nt)}$$

Con poco sforzo potremmo esplicitare $F_n(0)$, $F_n(\pi)$, $F_n(0)$.

(4) $\frac{d}{dt} v(x(t),t) = v_x \dot{x} + v_t \doteq v_x \cdot (1+x^2) + v_t$

$\Longleftrightarrow \begin{cases} \dot{x} = 1+x^2 \\ x(0) = x_0 \end{cases} \Longleftrightarrow \begin{cases} \frac{dx}{1+x^2} = dt \\ x(0) = x_0 \end{cases} \Longleftrightarrow t = \int_0^t ds = \int_{x(0)}^{x(t)} \frac{dy}{1+y^2} =$

$= \operatorname{arctg}(y) \Big|_{x_0}^{x(t)}$

$= \operatorname{arctg}(x(t)) - \operatorname{arctg}(x_0)$

$\Longleftrightarrow x(t) = \operatorname{tg}\big(t + \operatorname{arctg}(x_0)\big)$

Posto $\varphi(t) = v(x(t),t)$

$\varphi(0) = v(x(0),0) = v(x_0,0) = x_0$

abbiamo $\begin{cases} \dot{\varphi}(t) + t\varphi(t) = 0 \\ \varphi(0) = x_0 \end{cases} \Longleftrightarrow \begin{cases} e^{t^2/2}\dot{\varphi}(t) + e^{t^2/2} \cdot t \cdot \varphi(t) = 0 \\ \quad \| \\ \left[e^{t^2/2} \cdot \varphi(t)\right]^{\cdot} \\ \text{e } \varphi(0) = x_0 \end{cases}$

$\Longleftrightarrow e^{t^2/2}\varphi(t) = e^{0^2/2}\varphi(0) = x_0 \Longleftrightarrow \varphi(t) = e^{-t^2/2} \cdot x_0$

Quindi $v(x,t) = e^{-t^2/2} \cdot x_0$, ma $x_0 = \operatorname{tg}\big(\operatorname{arctg}(x(t)) - t\big)$,

$$\boxed{v(x,t) = e^{-t^2/2} \cdot \operatorname{tg}(\operatorname{arctg}(x) - t)}$$