Basic Functional Analysis.

L^2-spaces. $L^2(\mathbb{R}^n) \ni f$ if $f : \mathbb{R}^n \to \mathbb{R}$ and

$$\|f\|_{L^2}^2 := \int_{\mathbb{R}^n} |f(x)|^2 \, dx < \infty$$

Here the integral is in the sense of Lebesgue. We are interested in it only for its convergence properties:

(G) if $f_k \in L^2(\mathbb{R}^n)$ for $k \geq 0$ and

$$\lim_{k,\ell \to \infty} \|f_k - f_\ell\|_{L^2} = 0$$

then there is $f \in L^2(\mathbb{R}^n)$ s.t.

$$\lim_{k \to \infty} \|f_k - f\|_{L^2} = 0$$

We call f the L^2-limit of $\{f_k\}_{k\geq 0}$:

$$f = L^2\text{-}\lim_{k \to \infty} f_k$$

Examples show that the following can happen: $L^2\text{-}\lim_{k\to\infty} f_k = f$, but

$\forall x \in \mathbb{R}^n$: $\lim_{k \to \infty} f_k(x)$ does not exist.

$\|f\|_{L^2}$ should be interpreted as the norm of the "vector" f.

We have an inner product as well:

$$\langle f, g \rangle_{L^2} := \int_{\mathbb{R}^n} f(x)\, g(x) \, dx.$$

Observe that the usual properties of inner products hold:

(I1) $\langle f, g \rangle_{L^2} = \langle g, f \rangle_{L^2}$

(I2) $\langle \alpha f + \beta g, h \rangle_{L^2} = \alpha \langle f, h \rangle_{L^2} + \beta \langle g, h \rangle_{L^2}$ $\alpha, \beta \in \mathbb{R}$

(I3) $\langle f, f \rangle_{L^2} \geq 0$ $\forall f \in L^2$: we say that $f = 0$ in L^2 if $\|f\|_{L^2} = \langle f, f \rangle_{L^2}^{1/2} = 0$.

(I4) $|\langle f, g \rangle_{L^2}| \leq \|f\|_{L^2} \cdot \|g\|_{L^2}$

Property (I4) can be proved based on (I1)-(I3).

Properties (I1)-(I4) and the convergence property (G) are summarized by saying that $L^2(\mathbb{R}^n)$ is a Hilbert space.

If $f, g \in L^2$ and $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g \in L^2$ and

$$\|\alpha f\|_{L^2} = |\alpha| \cdot \|f\|_{L^2}$$

$$\|f + g\|_{L^2} \leq \|f\|_{L^2} + \|g\|_{L^2}$$

For $f, g \in L^2(\mathbb{R}^n)$,

$\|f-g\|_{L^2}$ measures the distance between f and g.

The Basic Structure of $L^2(\mathbb{R}^n)$.

f and $g \in L^2$ are orthogonal if $\langle f, g\rangle_{L^2} = 0$.

e.g. if $n \neq m$, then $f(x) = \sin(nx)$ and $g(x) = \sin(mx)$ are orthogonal in $L^2([-\pi, \pi])$:

$$\int_{-\pi}^{\pi} \sin(nx)\cdot \sin(mx)\, dx = 0.$$

A complete orthonormal system for $L^2(\mathbb{R}^n)$ is a sequence $\{e_k\}_{k=1}^{\infty}$ in $L^2(\mathbb{R}^n)$ s.t.

(i) $\|e_k\|_{L^2} = 1 \quad \forall k$ (normal)

(ii) $\langle e_k, e_h\rangle_{L^2} = 0 \quad \forall k \neq h$ (orthogonal)

(iii) for all $f \in L^2$, f can be written as

$$f = \sum_{k=1}^{\infty} \langle f, e_k\rangle_{L^2}\, e_k$$

and $$\|f\|_{L^2}^2 = \sum_{k=1}^{\infty} |\langle f, e_k\rangle_{L^2}|^2$$

The last assertion can be stated in the form of limit.

Let $f_\ell = \sum_{k=1}^{\ell} \langle f, e_k\rangle e_k$.

Then $\lim_{\ell\to\infty} \|f - f_\ell\|_{L^2} = 0$,

and $\|f - f_\ell\|_{L^2} = \left(\sum_{k=\ell+1}^{\infty} |\langle f, e_k\rangle|^2\right)^{1/2}$.

In applications, one finds a orthonormal system $\{e_k\}_{k=1}^{\infty}$, then one considers finite dimensional spaces

$$V_\ell = \{g = \sum_{k=1}^{\ell} \alpha_k e_k : \alpha_1, \dots, \alpha_\ell \in \mathbb{R}\}$$

The map $\sum_{k=1}^{\ell} \alpha_k e_k \longmapsto (\alpha_1, \dots, \alpha_\ell)$ identifies V_ℓ with $\mathbb{R}^\ell$, both as vector spaces and as spaces with inner products.

Example. Fourier system in $L^2([-\pi, \pi])$.

$e_n(x) = \frac{1}{\sqrt{\pi}} \sin(nx) \qquad n \geq 1$

$f_n(x) = \frac{1}{\sqrt{\pi}} \cos(nx) \qquad n \geq 1$

$f_0(x) = \frac{1}{\sqrt{2\pi}}$

$\{e_n\}_{n=0}^{\infty} \cup \{f_n\}_{n=1}^{\infty}$ is a o.n.s. for $L^2([-\pi, \pi])$.

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[a_n \cdot \frac{\cos(nx)}{\sqrt{\pi}} + b_n \cdot \frac{\sin(nx)}{\sqrt{\pi}}\right]$$

where
$$a_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \cos(ny) f(y)\, dy \quad (n \geq 1)$$

$$b_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} \sin(ny) f(y)\, dy \quad (n \geq 1)$$

$$a_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y)\, dy$$

Week convergence.

Let $\{f_k\}_{k=0}^{\infty}$ be a sequence in $L^2(\mathbb{R}^n)$.
We say that f_k converges to $f \in L^2(\mathbb{R}^n)$ weakly as $k \to \infty$ if $\forall g \in L^2(\mathbb{R}^n)$:

$$\lim_{k \to \infty} \langle f_k, g \rangle_{L^2} = \langle f, g \rangle_{L^2}.$$

We write $f_k \xrightarrow[k\to\infty]{} f$ or $w\text{-}\lim_{k \to \infty} f_k = g$

Obs. If $L^2\text{-}\lim_{k \to \infty} f_k = f$, then $w\text{-}\lim_{k\to\infty} f_k = f$:

$$|\langle f_k - f, g \rangle_{L^2}| \leq \|f_k - f\|_{L^2} \cdot \|g\|_2 \xrightarrow[k \to \infty]{} 0$$

$$|\langle f_k, g \rangle_{L^2} - \langle f, g \rangle_{L^2}| \qquad \text{if } f_k \xrightarrow[k\to\infty]{} f \text{ in } L^2$$

The converse is false. Let $\{e_k\}_{k=1}^{\infty}$ be a o.n.s. for $L^2(\mathbb{R}^n)$. Then

(i) $w\text{-}\lim_{k \to \infty} e_k = 0$, but

(ii) $\lim_{k\to\infty} \|e_k - 0\|_{L^2} = \lim_{k\to\infty} 1 = 1 \neq 0$.

It is still true that if $w\text{-}\lim_{k\to\infty} f_k = g$, then $\{f_k\}_{k=0}^{\infty}$ is bounded in $L^2(\mathbb{R}^n)$:

$\exists M \geq 0 \ \forall k \geq 0 \Rightarrow \|f_k\|_{L^2(\mathbb{R}^n)} \leq M$.

The main reason for caring about week convergence is the following compactness result (Banach-Alaoglu).

Theorem. If $\{f_k\}_{k \geq 0}$ is a sequence in $L^2(\mathbb{R}^n)$ such that $\|f_k\|_{L^2} \leq M < +\infty \ \forall k \geq 0$, then $\exists f \in L^2(\mathbb{R}^n)$, $\|f\|_{L^2} \leq M$, and a subsequence $\{f_{k_\lambda}\}_{\lambda \geq 0}$ of $\{f_k\}_{k \geq 0}$ s.t.

$$w\text{-}\lim_{\lambda \to \infty} f_{k_\lambda} = f.$$

The example $\{e_k\}_{k=1}^{\infty}$ seen before shows that the same is not true if we consider L^2-limits instead of week limits.

Sobolev Space

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, connected. Another space we are interested in is

$W^{1,2}(\Omega) = H^1(\Omega) = \{ f \in L^2(\Omega)$ such that for $j = 1, \ldots, n$ f has a weak derivative $\frac{\partial}{\partial x_j} f = g_j \in L^2(\Omega) \}$, i.e.

$$\forall \varphi \in C_0^1(\Omega) \Rightarrow \int_\Omega f \frac{\partial \varphi}{\partial x_j}\, dx = -\int_\Omega g_j \varphi\, dx.$$

Hilbert spaces. We need them because L^2, $W^{1,2}(\Omega)$ and other spaces of interest are Hilbert spaces.

H: vector space

$\forall u, v \in H \ \forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha u + \beta v \in \mathbb{R}$

and the usual algebra of vectors works.

$(u+v)+w = u+(v+w)$

$u+v = v+u$

$u+0 = u \qquad (\exists 0 \in H)$

$u+(-1)\cdot u = 0$

$(\alpha\beta)u = \alpha(\beta u) \qquad \alpha, \beta \in \mathbb{R}, \ u \in H$

$(\alpha+\beta)u = \alpha u + \beta u$

$\alpha(u+v) = \alpha u + \alpha v$

$1 \cdot u = u$

We have an inner product (scalar product):

$H \times H \ni (u, v) \longmapsto \langle u, v \rangle \in \mathbb{R}$.

with the usual properties:

$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle$

$\langle u, v \rangle = \langle v, u \rangle$

$\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$

Let $\|u\| = \langle u, u \rangle^{1/2}$. We have

$|\langle u, v \rangle| \leq \|u\| \cdot \|v\|$

The vector space $(H, +, \cdot)$ over $\mathbb{R}$ with the inner product $\langle \cdot, \cdot \rangle$ is a Hilbert space if it is complete:

if $v_k \in H \ \forall k \geq 0$ and

$$\lim_{k,h \to \infty} \|v_k - v_h\| = 0,$$

then there is $v \in H$ s.t.

$$\lim_{k \to \infty} v_k = v \text{ in } H,$$

i.e. $$\lim_{k \to \infty} \|v_k - v\| = 0.$$

Examples: (1) $\mathbb{R}^N$ is a (finite dimensional) Hilbert space if I let

$$\langle x, y \rangle = \sum_{j=1}^{N} x_j y_j$$

(2) $L^2(\mathbb{R}^N)$, $L^2(\Omega)$ are Hilbert spaces

(3) $W^{1,2}(\Omega)$ is a Hilbert space if

$$\langle f, g \rangle_{W^{1,2}(\Omega)} = \int_\Omega f g \, dx + \sum_{j=1}^{n} \int_\Omega \partial_j f \, \partial_j g \, dx$$

(this is far from obvious!).

(4) $W_0^{1,2}(\Omega) \subseteq W^{1,2}(\Omega)$: $f \in W_0^{1,2}(\Omega)$ iff there is a sequence $\{\varphi_k\}_{k=0}^{\infty}$ in $C_0^1(\Omega)$ s.t. $\|\varphi_k - f\|_{W^{1,2}(\Omega)} \xrightarrow[k \to \infty]{} 0$

is a Hilbert space: $\|f\|_{W_0^{1,2}(\Omega)} = \|f\|_{W^{1,2}(\Omega)}$.

Example (1) is finite dimensional.

There are vectors $e_k \in H = \mathbb{R}^N$ $(1 \le k \le N)$ such that

(a) $\langle e_k, e_h \rangle = \begin{cases} 0 & \text{if } k \ne h \\ 1 & \text{if } k = h \end{cases}$

(b) $\forall x \in \mathbb{R}^N$: $x = \sum_{k=1}^{N} x_k e_k$ with $x_1, \dots, x_N \in \mathbb{R}$

$x_k = \langle x, e_k \rangle$

Next best are examples (2), (3), (4): there are vectors $e_k \in H$ $(k = 1, \dots)$ such that (a) holds and

(b') $\forall v \in H \Rightarrow v = \sum_{k=1}^{\infty} v_k \cdot e_k$

with $v_k = \langle v, e_k \rangle \in \mathbb{R}$.

The equality means that

$$\lim_{\ell \to \infty} \left\| v - \sum_{k=1}^{\ell} v_k e_k \right\| = 0.$$

In this case we say that H is separable. (All our spaces are infinite-dimensional, but not separable).

The norm $\|\cdot\|$ has the usual properties:

$\|v\| \ge 0$ and $\|v\| = 0 \Leftrightarrow v = 0$

$\|\alpha v\| = |\alpha| \cdot \|v\|$ if $\alpha \in \mathbb{R}$ and $v \in H$

$\|V + v\| \le \|V\| + \|v\|$.

If H is a separable Hilbert space and $\{e_k\}_{k=1}^{\infty}$ is a o.n.s. for it, consider the spaces $V_N \subseteq H$ $(N \ge 1)$,

$V_N = \operatorname{span}\{e_1, \dots, e_N\} = \{ \sum_{j=1}^{N} \alpha_j e_j : \alpha_1, \dots, \alpha_N \in \mathbb{R} \}$.

V_N is a finite dimensional Hilbert space: $\dim(V_N) = N$

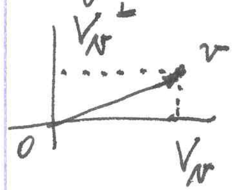
and $v = \sum_{j=1}^{N} \langle v, e_j \rangle e_j \mapsto (\langle v, e_1 \rangle, \dots, \langle v, e_N \rangle)$

identifies V_N with $\mathbb{R}^N$ in all respects.

This is the basis of all Fourier analysis and more (Wavelets, MRA, orthogonal polynomials, Bessel functions ...).

[Develop the Fourier series case].

In general we can project $v \in H$,



$v = \sum_{k=1}^{\infty} v_k e_k$

onto V_N:

$v = P_N v + Q_N v$

$V_N \ni P_N v = \sum_{k=1}^{N} v_k e_k$, $Q_N v = \sum_{k=N+1}^{\infty} v_k e_k \in V_N^\perp$:

$\langle P_N v, Q_N v \rangle = 0$ and $\|v\|^2 = \|P_N v\|^2 + \|Q_N v\|^2$.

Given a sequence $\{v^k\}_{k=0}^{\infty}$ in H,

the <u>weak limit</u> of v^k is $v \in H$,

$\underset{k\to\infty}{w\text{-}\lim}\, v^k = v$ if $\forall u \in H: \langle v^k - v, u\rangle \underset{k\to\infty}{\longrightarrow} 0$

(1) $\lim_{k\to\infty} v^k = v$ in $H \Rightarrow \underset{k\to\infty}{w\text{-}\lim}\, v^k = v$

(2) $\underset{k\to\infty}{w\text{-}\lim}\, v^k = v \Rightarrow \{v^k\}$ is bounded in H:

$\|v^k\| \le M \quad \forall k \ge 0.$

(3) if $\{v^k\}_{k=0}^{\infty}$ is bounded in H,

then is a subsequence

$\{v^{k_\ell}\}_{\ell=0}^{\infty}$ s.t. $\underset{\ell\to\infty}{w\text{-}\lim}\, v^{k_\ell} = v \in H$ exists.

(<u>Banach-Alaoglu Theorem</u>).