$z = x+iy$ $\qquad \mathbb{C} \supseteq \Omega \xrightarrow{f} \mathbb{C}$ $\qquad \Omega$ open

$f = u+iv$ $\qquad z \longmapsto f(z) = w$

We have checked that:

Theorem. f is holomorphic in $\Omega \iff f$ satisfies the Cauchy-Riemann equations:

$$\text{C.R.}\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

In the course of proof we have seen that, if F is holomorphi[c] then

$$f'(z) = \frac{\partial}{\partial x} f(z) = u_x(z) + i\, v_x(z)$$
$$= \frac{1}{i}\frac{\partial}{\partial y} f(z) = v_y(z) - i\, u_y(z)$$

Exercises.

(1) Let $f(z) = e^z$; $f: \mathbb{C} \to \mathbb{C}$.
Recall that $e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$.
Verify that f is holomorphic and find $F'(z)$.

(2) For $z = re^{i\theta}$, $\pi < \theta < \pi$, define
$g(z) = \log(z) = \log r + i\theta$
Is g holomorphic in $\Omega = \{z = re^{i\theta}: r>0;\ -\pi < \theta < \pi\}$?
If such is the case, compute $g'(z)$.

(3) For $f(z) = u(z) + i\,v(z)$ given below, compute
$J\begin{pmatrix} u & v \\ x & y\end{pmatrix} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ and decide if f is holomorphi[c]

(i) $f(z) = |z|^2$

(ii) $f(z) = z\bar{z}^2 - 3i\bar{z} + 1$

(iii) $f(z) = \bar{z}^2$

(iv) $f(z) = z^2$

(4) Show that if $z \in \mathbb{C}$, then for $n \in \mathbb{N}$:

$$(1 + z + z^2 + \cdots + z^n)(1 - z) = 1 - z^{n+1}.$$

Deduce that for $z \in \mathbb{C}$, $|z| < 1$, one has:

$$\lim_{n \to \infty} \sum_{j=0}^{n} z^j = \frac{1}{1 - z}$$

Notation: we write $\lim_{n \to \infty} \sum_{j=0}^{n} z^j = \sum_{j=0}^{\infty} z^j$

if the limit exists in $\mathbb{C}$.

(5) Find the solutions of

$$z^3 + z^2 + z + 1 = 0$$

in $\mathbb{C}$ and represent them on the complex plane.

Hint. We know everything on $z^4 = 1$ and so ...

(6) Find the solutions of

$$z^4 + z^2 + 1 = 0$$

in $\mathbb{C}$ and represent them on the complex plane.

(7) Do the same with $z^4 - z^2 + 1 = 0$

(8) Compute $1 + e^{2\pi i\, 1/5} + e^{2\pi i\, 2/5} + e^{2\pi i\, 3/5} + e^{2\pi i\, 4/5}$

Hint. Let $z = e^{2\pi i\, 1/5}$.