The goal of the following exercise is showing that if $z \in \mathbb{C}$, then

$$(*)\ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

where the factorial $n!$ is defined by

$$\begin{cases} 0! = 1 \\ n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1 \text{ if } n \geq 1. \end{cases}$$

**(1)**Line integrals.

(i) Let $\Gamma_R = \{\zeta \in \mathbb{C} : |\zeta| = R\}$ be the circle centered at 0 and having radius $R > 0$, with a counterclockwise parametrization. Show that

$$e^z = \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_{\Gamma_R} \frac{e^\zeta}{\zeta^{n+1}} d\zeta \ \cdot \ z^n.$$

(ii) Show that, if $n \geq 0$, then

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{e^\zeta}{\zeta^{n+1}} d\zeta = \frac{1}{2\pi i n} \int_{\Gamma_R} \frac{e^\zeta}{\zeta^n} d\zeta.$$

**Hint.** Make the line integral explicit and use integration by parts.

(iii) Show that

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{e^\zeta}{\zeta^{n+1}} d\zeta = \frac{1}{n!}.$$

Deduce from this the series expansion of $e^z$.

**(2)**Let $\Gamma_R = \{\zeta \in \mathbb{C} : |\zeta| = R\}$ as above, with a counterclockwise parametrization. Compute, for $R > 0$, $R \neq 1$, the line integral

$$I(R) := \int_{\Gamma_R} \frac{z^3 - 2z}{z - i} dz.$$

**Hint.** Use Cauchy Theorem and Cauchy formual. You should obtain two values of $I(R)$, depending on $R$ being in $(0, 1)$ or in $(1, \infty)$.