Mathematical Methods - Mathematical Analysis: 1^{st} midterm (08/01/2014)

Name...Family name.. University ID number.................

1 Find the conditions on a, b, c, d, such that

$$f(z) = a \cdot \frac{z}{|z|^2} + b \cdot \frac{\overline{z}}{|z|^2} + c \cdot z + d \cdot \overline{z}.$$

defines a function holomorphic in $\{z \in \mathbb{C} : \ z \neq 0\}$.

2 Let $-1 < a, b < 1$. Compute the integral:

$$I_{a,b} = \int_{-\pi}^{+\pi} \frac{dt}{|e^{it} - a|^2 \cdot |e^{it} - b|^2},$$

and use the result to compute

$$I_{a,a} = \int_{-\pi}^{+\pi} \frac{dt}{|e^{it} - a|^4}.$$

MH.1

MIDTERM

① $f(z) = a\dfrac{z}{|z|^2} + b\dfrac{\bar z}{|z|^2} + cz + d\bar z$

$= a\dfrac{z}{z\bar z} + b\dfrac{\bar z}{z\bar z} + cz + d\bar z$

$= \dfrac{a}{\bar z} + d\bar z + \dfrac{b}{z} + c\cdot z = h(\bar z) + g(z)$

Since $g(z) = \dfrac{b}{z} + c\cdot z$ is holomorphic in $\mathbb{C}\setminus\{0\}$,

then f is holomorpic in $\mathbb{C}\setminus\{0\} \Leftrightarrow f(z) - g(z) = h(\bar z) = \dfrac{a}{\bar z} + d\bar z$ is holomorphic in $\mathbb{C}\setminus\{0\}$. Now

$h(\bar z) = h(x-iy) = \dfrac{a}{x-iy} + d(x-iy) = a\dfrac{x+iy}{x^2+y^2} + d(x-iy)$

$= \left(\dfrac{ax}{x^2+y^2} + dx\right) + i\left(\dfrac{ay}{x^2+y^2} - dy\right) = u + iv.$

$u_x = a\dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2} + d \qquad u_y = -\dfrac{2axy}{(x^2+y^2)^2}$

$v_x = -\dfrac{2axy}{(x^2+y^2)^2} \qquad v_y = a\cdot\dfrac{x^2+y^2-2y^2}{(x^2+y^2)^2} - d$

$u_y = -v_x \Leftrightarrow a = 0$, in which case $u_x = d$, $v_y = -d$: $u_x = v_y \Leftrightarrow d = 0$

i.e. $\boxed{f \text{ is holomorphic} \Leftrightarrow a = d = 0 \Leftrightarrow f(z) = \dfrac{b}{z} + cz}$

② $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \dfrac{dt}{|e^{it}-a|^2\cdot|e^{it}-b|^2} =$ $\qquad z = e^{it}$, $dz = i e^{it}\,dt = iz\,dt$

$= \dfrac{1}{2\pi i}\displaystyle\int_{\{z:|z|=1\}} \dfrac{dz}{|z-a|^2\cdot|z-b|^2\cdot z} = \dfrac{1}{2\pi i}\int_{\{\cdot\}} \dfrac{dz}{(z-a)(\bar z-a)(z-b)(\bar z-b)z}$

$= \dfrac{1}{2\pi i}\displaystyle\int_{\{\cdot\}} \dfrac{dz}{(z-a)\left(\frac{1}{z}-a\right)(z-b)\left(\frac{1}{z}-b\right)z} = \dfrac{1}{2\pi i}\int_{\{\cdot\}} f(z)\,dz$

with $f(z) = \dfrac{z}{(z-a)(1-az)(z-b)(1-bz)}$: singularities of 1st order at $z = a, b$ (in $\Delta = \{z : |z|<1\}$) and $1/a, 1/b \notin \Delta$

$= \displaystyle\sum_{\substack{w:\text{ singularity} \\ \text{of } f \text{ in } \{z:|z|<1\} = \Delta}} \operatorname{Res}(f, w) = \operatorname{Res}(f, a) + \operatorname{Res}(f, b)$

$= \dfrac{a}{(1-a^2)(a-b)(1-ab)} + \dfrac{b}{(b-a)(1-ab)(1-b^2)}$

HM2

$$= \frac{a(1-b^2) - b(1-a^2)}{(1-a^2)(1-ab)(a-b)} = \frac{a - b - ab^2 + a^2 b}{(1-a^2)(1-ab)(a-b)}$$

$$= \frac{(a-b)(1+ab)}{(1-a^2)(1-ab)(a-b)} = \frac{1+ab}{(1-a^2)(1-b^2)}$$

Then $\boxed{I_{a,b} = 2\pi \cdot \frac{1+ab}{(1-a^2)\cdot(1-b^2)}}$

and $\boxed{I_{a,a} = \lim_{b \to a} I_{a,b} = 2\pi \cdot \frac{1+a^2}{(1-a^2)^2}}$

Mathematical Methods - Mathematical Analysis: takehome (simulation)

Suppose $T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ is defined as

$$\widehat{Ta}(t) = \begin{cases} 0 \text{ if } -\pi \leq t \leq 0, \\ \left(\frac{\pi}{2} - \left|t - \frac{\pi}{2}\right|\right) \cdot \hat{a}(t) \text{ if } 0 < t \leq \pi. \end{cases}$$

for all $a \in \ell^2(\mathbb{Z})$.

- Is T linear?
- Is T stable?
- Is T time-invariant?
- Is T realizable?

Motivate your answer.

Find $b \in \ell^2(\mathbb{Z})$ such that

$$Ta(n) = \sum_{k=-\infty}^{+\infty} a(n-k)b(k).$$

Compute $T\delta_b$, where b is the last digit of your student ID.

It is useful to know that:

- $\hat{a}(t) = \sum_{k=-\infty}^{+\infty} a(k)e^{-ikt}$;
- $\delta_k : \mathbb{Z} \to \mathbb{C}$ is a discrete Dirac's delta, defined as

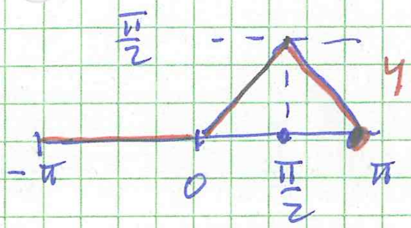
$$\delta_k(n) = \begin{cases} 1 \text{ if } n = k, \\ \hat{a}(t) \text{ if } n \neq k. \end{cases}$$

- for the meaning of linear, stable, time-invariant and realizable see your notes from the class, or the solution to the exercise.

$T: \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \qquad (Ta)^\wedge(t) = \hat{a}(t)\, m(t) \quad \text{if } |t| \le \pi$

TAKE-HOME

with
$$m(t) = \begin{cases} 0 & \text{if } -\pi \le t \le 0 \\ \frac{\pi}{2} - \left|t - \frac{\pi}{2}\right| & \text{if } 0 \le t \le \pi \end{cases}$$



T is linear: $\widehat{T(\lambda a + \mu b)}(t) =$

$= m(t)\, \widehat{\lambda a + \mu b}(t) = m(t)\left(\lambda \hat{a}(t) + \mu \hat{b}(t)\right)$

$= \lambda \cdot m(t) \hat{a}(t) + \mu \cdot m(t) \hat{b}(t) = \lambda \cdot \widehat{Ta}(t) + \mu \cdot \widehat{Tb}(t)$

For $\lambda, \mu \in \mathbb{C}$; $a, b \in \ell^2(\mathbb{Z}) \Rightarrow T(\lambda a + \mu b) = \lambda \cdot Ta + \mu \cdot Tb$

T is stable: $|m(t)| \le \pi/2 \quad \forall t \in [-\pi, \pi]$.

IN PARTICULAR, $\|Ta\|_{\ell^2(\mathbb{Z})} \le \frac{\pi}{2} \cdot \|a\|_{\ell^2(\mathbb{Z})}$.

T is time-invariant since $(Ta)^\wedge(t) = \hat{a}(t)\, m(t)$.

Moreover, $Ta = (T\delta_0) * a$ where $(T\delta_0)^\wedge(t) = m(t)$.

IN THE TEXT OF THE EXERCISE I CALL $b = T\delta_0$:

$$b(n) = \check{m}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} m(t) e^{itn}\, dt = \frac{1}{2\pi}\int_0^{\pi} \left(\frac{\pi}{2} - \left|t - \frac{\pi}{2}\right|\right) e^{itn}\, dt$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi/2} t e^{itn}\, dt + \int_{\pi/2}^{\pi} (\pi - t) e^{itn}\, dt\right) = \begin{cases} (\pi/2)^2/2\pi = \pi/8 & \text{if } n = 0 \\ \to ⓪ & \end{cases}$$

$$⓪ \to = \frac{1}{2\pi}\left\{\left[\frac{t e^{itn}}{in}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{e^{itn}}{in}\, dt + \left[\frac{(\pi - t) e^{itn}}{in}\right]_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{e^{itn}}{in}\, dt\right\}$$

$$= \frac{1}{2\pi i n}\left\{\cancel{\frac{\pi}{2} e^{i\frac{\pi}{2}n}} - \left[\frac{e^{itn}}{in}\right]_0^{\pi/2} - \cancel{\frac{\pi}{2} e^{i\frac{\pi}{2}n}} + \left[\frac{e^{itn}}{in}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{2\pi (in)^2} \cdot \left(1 + e^{i\pi n} - 2 e^{i\pi/2 n}\right)$$

$$= -\frac{1}{2\pi n^2} \cdot \left(1 + (-1)^n - 2 i^n\right) = \begin{cases} n = 2m+1 \text{ odd: } \dfrac{i(-1)^m}{\pi (2m+1)^2} \\ n = 2m \text{ even: } -\dfrac{(1 - (-1)^m)}{4\pi m^2} \end{cases}$$

$$\boxed{b(n) = \begin{cases} \pi/8 & \text{if } n = 0 \\ \dfrac{i(-1)^m}{\pi (2m+1)^2} & \text{if } n = 2m+1 \\ -\dfrac{1 - (-1)^m}{[illegible]} & \text{if } n = 2m,\ m \ne 0 \end{cases}}$$

T is not realizable since $b(n) \ne 0$ for some $n < 0$.

THE LAST DIGIT OF MY FACULTY ID IS $k=1$:

T4.

$$Ta(n) = a(n)\cdot\frac{\pi}{8} - \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} \frac{a(n-2m)}{4\pi m^2}\left[1-(-1)^m\right] + \frac{i}{\pi}\sum_{m=-\infty}^{+\infty} \frac{a(n-2m-1)(-1)^m}{(2m+1)^2}$$

ANT

$$\boxed{T\delta_1(n) = \begin{cases} \pi/8 & \text{iF } n=1 \\ -\dfrac{\left[1-(-1)^m\right]}{4\pi m^2} & \text{iF } n=2m+1 \ (m\in\mathbb{Z}, m\neq 0) \\ \dfrac{i}{\pi}(-1)^m\dfrac{1}{(2m+1)^2} & \text{iF } n=2m+2 \end{cases}}$$