Mathematical Methods - Mathematical Analysis: 1^{st} midterm (20/12/2013)

Name..Family name.. University ID number..................

1 Find real k such that

$$f(x+iy) = x\arctan(y/x) + y\log\sqrt{x^2+y^2} - 7ki(x\log\sqrt{x^2+y^2} - y\arctan(y/x))$$

defines a function holomorphic in $\{z = x+iy : \ y > 0\}$.

2 Compute the integral:

$$\int_{-\infty}^{+\infty} \frac{1-x}{1+7^2x^2+7^4x^4}dt.$$

Hint. $(u-1)(u^2+u+1) = u^3-1$.

① $f = u + iv$

$$u_x = \arctan(y/x) + x \cdot \frac{-y/x^2}{1+(y/x)^2} + y \cdot \frac{1}{2} \frac{2x}{x^2+y^2}$$

$$= \arctan(y/x) + \frac{-xy+xy}{x^2+y^2} = \arctan(y/x)$$

$$u_y = \frac{x \cdot 1/x}{1+(y/x)^2} + \log\sqrt{x^2+y^2} + y \cdot \frac{1}{2} \cdot \frac{2y}{x^2+y^2}$$

$$= \frac{x^2+y^2}{x^2+y^2} + \log\sqrt{x^2+y^2} = 1 + \log\sqrt{x^2+y^2}$$

$$v_x = -7\cancel{0}k \cdot \left\{ \log\sqrt{x^2+y^2} + x \frac{1}{2} \frac{2x}{x^2+y^2} - y \cdot \frac{-y/x^2}{1+(y/x)^2} \right\}$$

$$= -7\cancel{0}k \cdot \left\{ \log\sqrt{x^2+y^2} + \frac{x^2+y^2}{x^2+y^2} \right\} = -7\cancel{0}k \cdot \left\{ \log\sqrt{x^2+y^2} + 1 \right\}$$

$$v_y = -7\cancel{0}k \cdot \left\{ x \cdot \frac{1}{2} \cdot \frac{2y}{x^2+y^2} - \arctan(y/x) - y \frac{1/x}{1+(y/x)^2} \right\}$$

$$= -7\cancel{0}k \cdot \left\{ \frac{xy - xy}{x^2+y^2} - \arctan(y/x) \right\} = 7\cancel{0}k \cdot \arctan(y/x)$$

$$u_x = v_y \iff 7k = 1 \iff \boxed{k = 1/7} \implies u_y = -v_x.$$

② $$\int_{-\infty}^{+\infty} \frac{1-x}{1+7^2 \cdot x^2 + 7^4 \cdot x^4} \, dx = \int_{-\infty}^{+\infty} \frac{dx}{1+7^2 \cdot x^2 + 7^4 \cdot x^4} \doteq$$

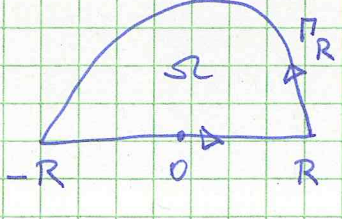
because $\varphi(x) = \dfrac{x}{1+7^2 \cdot x^2 + 7^4 \cdot x^4}$ is odd $(-\varphi(x) = \varphi(-x))$,

hence $\displaystyle\int_{-\infty}^{+\infty} \varphi(x) \, dx = 0$

$$\doteq \frac{1}{7} \cdot \int_{-\infty}^{+\infty} \frac{dy}{1+y^2+y^4}$$ after setting $y = 7x$, $dy = 7 \cdot dx$

$$= \frac{1}{7} \lim_{R\to\infty} I_R \quad \text{with } I_R = \int_{-R}^{R} \frac{dy}{1+y^2+y^4}.$$

Let $$f(z) = \frac{1}{1+z^2+z^4} = \frac{z^2-1}{(z^2-1)(z^4+z^2+1)} = \frac{z^2-1}{z^6-1}$$



$$\boxed{\int_{-R}^{R} f(x) \, dx + \int_{\Gamma_R} f(z) \, dz = 2\pi i \cdot \sum_{w \in \Omega} \operatorname{Res}(f, w)}$$

$$\left|\int_{\Gamma_R} f(z)\,dz\right| \underset{\substack{\uparrow\\ z=Re^{i\theta}}}{=} \left|\int_0^{\pi} \frac{iRe^{i\theta}\,d\theta}{1+(Re^{i\theta})^2+(Re^{i\theta})^4}\right|$$

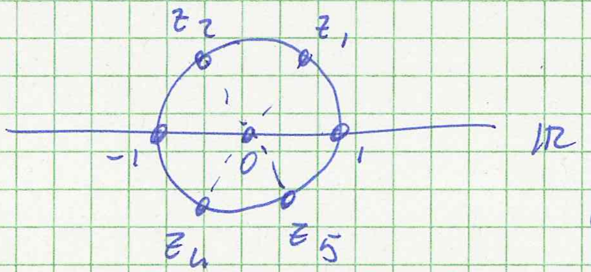
$$\underset{\uparrow}{\leq} \frac{R}{R^4-R^2-1}\int_0^{\pi} d\theta = \frac{\pi\cdot R}{R^4-R^2-1} \xrightarrow[R\to+\infty]{} 0$$

iF R is large enough so that $R^4-R^2-1>0$

THEN $\int_{\Gamma_R} f(z)\,dz \xrightarrow[R\to+\infty]{} 0$; HENCE,

$$\boxed{\lim_{R\to+\infty}\left(\int_{\Gamma_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx\right) = \int_{-\infty}^{+\infty} f(x)\,dx.}$$

THE SINGULARITIES OF f ARE z: $z^6-1=0$, $z^2-1\neq 0$



$$z_j = e^{2\pi i\cdot j/6} = e^{\frac{\pi}{3}i\cdot j}$$

$j = 1, 2, 4, 5$

ONLY $z_1, z_2 \in \Omega$ (iF $R>1$).

$$f(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_4)(z-z_5)}$$

$$\Rightarrow \int_{-\infty}^{+\infty} f(x)\,dx = \mathrm{Res}(f,z_1)+\mathrm{Res}(f,z_2) =$$

$$= \frac{1}{(z_1-z_2)(z_1-z_4)(z_1-z_5)} + \frac{1}{(z_2-z_1)(z_2-z_4)(z_2-z_5)} \overset{\bullet}{=}$$

$$\boxed{\begin{array}{ll} z_1 = \cos(\pi/3)+i\cdot\sin(\pi/3) = \frac{1}{2}+i\frac{\sqrt{3}}{2} & z_2 = -\frac{1}{2}+i\frac{\sqrt{3}}{2} \\ z_4 = -z_1 = -\frac{1}{2}-i\cdot\frac{\sqrt{3}}{2} & z_5 = -z_2 = +\frac{1}{2}-i\frac{\sqrt{3}}{2} \end{array}}$$

$$\overset{\bullet}{=} \frac{1}{1\cdot 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\cdot 2i\frac{\sqrt{3}}{2}} + \frac{1}{(-1)\cdot 2i\frac{\sqrt{3}}{2}\cdot 2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{i\sqrt{3}}\,\frac{(1-i\sqrt{3})+1+i\sqrt{3}}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{2}{i\sqrt{3}\cdot 4} = \frac{1}{2i\sqrt{3}}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1-x}{1+7^2x^2+7^4x^4}\,dx = \frac{1}{7}\cdot 2\pi i\cdot\frac{1}{2i\sqrt{3}} = \boxed{\frac{\pi}{7\cdot\sqrt{3}}}$$