Let $\varphi : [-\pi,\pi] \to \mathbb{C}$

$$\varphi(t) = \begin{cases} -1 & \text{if } -\pi \le t \le 0 \\ +1 & \text{if } 0 < t \le \pi \end{cases}$$

THEN $\widehat{Ta}(t) = \hat{a}(t) \cdot \varphi(t)$.

ALSO $|\varphi(t)| \le 1 \quad \forall t \in [-\pi,\pi]$.

THEN φ is BOUNDED AND

$$\ell^2(\mathbb{Z}) \xrightarrow{T} \ell^2(\mathbb{Z})$$

IS LINEAR, STABLE (BECAUSE φ IS BOUNDED) AND TIME-INVARIANT.

MOREOVER

$$Ta = a * b$$

WHERE $b = T\delta_0$ AND $\hat{b}(t) = \varphi(t) \quad \forall t \in [-\pi,\pi]$

I HAVE TO COMPUTE

$$b(n) = \check{\varphi}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)\, e^{int}\, dt$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} e^{int}\, dt - \int_{0}^{\pi} e^{int}\, dt \right)$$

$$= \frac{1}{2\pi} \left\{ \left[\frac{e^{int}}{in} \right]_{t=-\pi}^{t=0} - \left[\frac{e^{int}}{in} \right]_{t=0}^{t=\pi} \right\}$$

IF $n \neq 0$

$$= \frac{1}{2\pi i n} \cdot \left[\left(1 - e^{-in\pi}\right) - \left(e^{in\pi} - 1\right) \right]$$

$$= \frac{1}{2\pi i n} \cdot \left[2 - 2(-1)^n \right]$$

$$e^{in\pi} = \left(e^{i\pi}\right)^n = (-1)^n = \left(e^{-i\pi}\right)^n = e^{-in\pi}$$

$$= \frac{1}{\pi i n} \left[1 - (-1)^n \right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{2}{\pi i n} & \text{if } n \text{ is odd} \end{cases}$$

ALSO, $b(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t)\, dt = 0$

THEN:

$$b(n) = \begin{cases} 0 & \text{if } n \in \mathbb{Z},\ n \text{ EVEN} \\ \frac{2}{\pi i n} & \text{if } n \in \mathbb{Z},\ n \text{ ODD} \end{cases}$$

T IS NOT REALIZABLE BECAUSE THERE ARE $n \in \mathbb{Z}$ SUCH THAT $n < 0$ AND $b(n) \neq 0$.

$$Ta(n) = \sum_{k=-\infty}^{+\infty} a(n-k)\, b(k) =$$

$\boxed{k \text{ is odd} \iff k = 2h+1 \text{ FOR SOME } h \in \mathbb{Z}}$

$$= \cancel{\sum_{h=-\infty}^{+\infty} a(n-k) \cdot \frac{2}{\pi i}}$$

$$= \sum_{h=-\infty}^{+\infty} a(n-2h-1)\, \frac{2}{\pi i (2h+1)}$$

THEN:
$$T\delta_3(n) = \begin{cases} \frac{2}{\pi i (n-3)} & \text{if } n \text{ is EVEN} \\ 0 & \text{if } n \text{ is ODD} \end{cases}$$