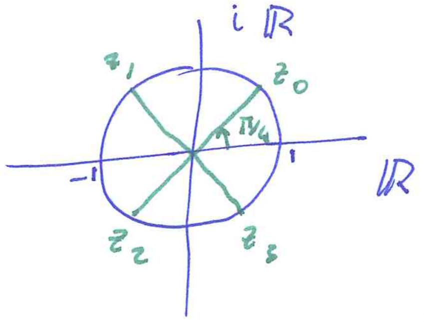
Integrali con residui di funzioni razionali

(1) $I = \int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx$ $\qquad f(z) = \frac{z^2}{z^4+1}$

$z^4 = -1 = e^{i\pi} \iff z = z_k = e^{i\frac{\pi}{4} + ik\frac{\pi}{2}}$ con $k = 0, 1, 2, 3$

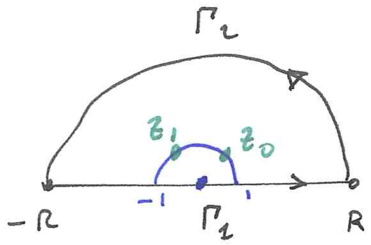


$$\mathrm{Res}(f, z_k) = \lim_{z\to z_k} (z - z_k)\frac{z^2}{z^4+1}$$

$$= z_k^2 \cdot \lim_{z\to z_k} \frac{(z - z_k)}{z^4+1} =$$

$$= z_k^2 \cdot \lim_{z\to z_k} \frac{1}{4\cdot z^3} \quad \text{(de l'Hôpital)}$$

$$= \frac{1}{4 z_k}.$$



Fisso $R > 1$ e ho che, per il teorema dei residui:

$$\frac{1}{4z_0} + \frac{1}{4z_1} = \mathrm{Res}(f, z_0) + \mathrm{Res}(f, z_1) = \frac{1}{2\pi i}\int_{\Gamma_1 + \Gamma_2} f(z)\,dz$$

$$= \frac{1}{2\pi i}\int_{-R}^{R} \frac{x^2}{x^4+1}\,dx + \frac{1}{2\pi i}\int_0^{\pi} \frac{(Re^{i\theta})^2\, R i e^{i\theta}\,d\theta}{(Re^{i\theta})^4+1} = A + B$$

Ho che $A \xrightarrow[R\to\infty]{} \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{x^2}{x^4+1}\,dx = \frac{I}{2\pi i}$

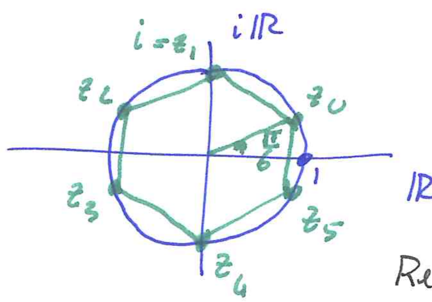
e che $|B| \le \int_{0}^{\pi} \frac{R^3}{R^4-1}\,d\theta = \pi \cdot \frac{R^3}{R^4-1} \xrightarrow[R\to\infty]{} 0$,

quindi $\frac{I}{2\pi i} + 0 = \frac{1}{4}\left(\frac{1}{z_0} + \frac{1}{z_1}\right) = \frac{1}{4}\left(\bar{z}_0 + \bar{z}_1\right) = \frac{z_3 + z_2}{4} = -\frac{i\cdot 2}{4\sqrt{2}}$

cioè $I = 2\pi i \cdot \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}.$

(2) La linea generale del ragionamento è la stessa di (1), come negli esercizi che seguono. Vediamo solo quali dettagli cambiano.

$f(z) = \frac{z^4}{z^6+1}$; $\quad z^6 = -1 = e^{\pi i} \iff z = z_k = e^{\frac{\pi i}{6} + k\frac{\pi i}{3}}$; $k = 0, 1, 2, 3, 4, 5$.



$z_0 = e^{i\pi/6} = \frac{\sqrt{3}}{2} + \frac{i}{2}$ $\qquad z_1 = i$

$z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$ $\qquad z_3 = -\frac{\sqrt{3}}{2} - \frac{i}{2}$

$z_4 = -i$ $\qquad z_5 = \frac{\sqrt{3}}{2} - \frac{i}{2}$

$$\mathrm{Res}(f, z_k) = \lim_{z\to z_k} \frac{(z - z_k)\cdot z^4}{z^6+1} = \frac{z_k^4}{5 z_k^5} = \frac{1}{5 z_k}.$$

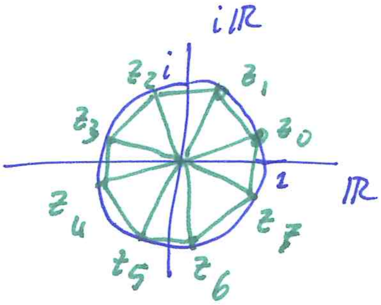
$$\int_{-\infty}^{+\infty} \frac{x^4}{x^6+1}\,dx = 2\pi i\cdot\left[Res(f,z_0)+Res(f,z_1)+Res(f,z_2)\right]$$

$$= \frac{2\pi i}{5}\cdot\left(\frac{1}{z_0}+\frac{1}{z_1}+\frac{1}{z_2}\right) = \frac{2\pi i}{5}\cdot(z_5+z_4+z_3)$$

$$= \frac{2\pi i}{5}\cdot(-2i) = \frac{4}{5}\pi.$$

(3) $f(z)=\dfrac{1}{z^8+1}$ $\qquad z^8=-1=e^{i\pi} \Leftrightarrow z=z_k = e^{\frac{i\pi}{8}+\frac{ki\pi}{4}}$; $k\in\mathbb{N}$, $0\le k\le 7$.

$$\int_{-\infty}^{+\infty} \frac{dx}{x^8+1} = 2\pi i\cdot\sum_{k=0}^{3} Res(f,z_k) = 2\pi i\cdot\sum_{k=0}^{3}\frac{1}{8\cdot z_k^7} \overset{*}{=}$$



$$= \frac{2\pi i}{8}\cdot\left(\frac{1}{z_3}+\frac{1}{z_2}+\frac{1}{z_1}+\frac{1}{z_0}\right) =$$

$$= \frac{\pi i}{4}(z_4+z_5+z_7+z_6)$$

$$= \frac{\pi i}{4}\cdot\left[(z_4+z_7)+(z_5+z_6)\right] =$$

$*$ $z_0^7=z_3$
$z_1^7=z_2$
$z_2^7=z_1$
$z_3^7=z_0$

$$= \frac{\pi i}{4}\cdot\left(-2i\sin(\pi/8)-2i\sin\left(\tfrac{3}{8}\pi\right)\right) = \frac{\pi}{2}\cdot\left[\sin\left(\tfrac{\pi}{8}\right)+\sin\left(\tfrac{3}{8}\pi\right)\right]$$

Se voglio un risultato più algebrico, osservo che $\frac{\pi}{8}+\frac{3}{8}\pi=\frac{\pi}{2}$,

quindi $\sin\left(\frac{3}{8}\pi\right)=\cos\left(\frac{\pi}{8}\right)$ e

$$\sin\left(\tfrac{\pi}{8}\right)+\cos\left(\tfrac{\pi}{8}\right)=\sqrt{\left(\sin\left(\tfrac{\pi}{8}\right)+\cos\left(\tfrac{\pi}{8}\right)\right)^2}=\sqrt{1+2\cdot\sin\left(\tfrac{\pi}{8}\right)\cos\left(\tfrac{\pi}{8}\right)}$$

$$= \sqrt{1+\sin(\pi/4)} = \sqrt{1+\frac{1}{\sqrt{2}}}$$

Da cui $\displaystyle\int_{-\infty}^{+\infty}\frac{dx}{x^8+1} = \frac{\pi}{2}\cdot\sqrt{1+\frac{1}{\sqrt{2}}}.$

(4) Come in (3), ma questa volta $\displaystyle\int_{-\infty}^{+\infty}\frac{x^2}{x^8+1}\,dx =$

$$= 2\pi i\cdot\sum_{k=0}^{3}\frac{z_k^2}{8z_k^7} = \frac{\pi i}{4}\left(\frac{1}{z_0^5}+\frac{1}{z_1^5}+\frac{1}{z_2^5}+\frac{1}{z_3^5}\right) =$$

$$= \frac{\pi i}{4}\cdot\left(\frac{1}{z_2}+\frac{1}{z_7}+\frac{1}{z_4}+\frac{1}{z_1}\right) = \frac{\pi i}{4}\cdot(z_5+z_0+z_3+z_6)$$

$$= \frac{\pi i}{4}\left[-2i\sin\left(\tfrac{3}{8}\pi\right)+2i\sin\left(\tfrac{\pi}{8}\right)\right] = \frac{\pi}{2}\cdot\left[\sin\left(\tfrac{3}{8}\pi\right)-\sin\left(\tfrac{\pi}{8}\right)\right]$$

$$= \frac{\pi}{2}\left[\cos\left(\tfrac{\pi}{8}\right)-\sin\left(\tfrac{\pi}{8}\right)\right] = \frac{\pi}{2}\cdot\sqrt{1-\frac{1}{\sqrt{2}}}.$$

(5) Sia $\Delta = \Delta(0,1)$ il disco unitario in $\mathbb{C}$ e sia f olomorfa in Ω dove $\Omega \supsetneq \bar{\Delta} = \Delta \cup \partial\Delta$.

(i) Mostrare che se $\mathbb{T}$ è il bordo di Δ orientato in senso antiorario, allora, se $w \in \Delta$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\theta})\, d\theta}{1 - w \cdot e^{-i\theta}} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{1-\bar{z}w}\, \frac{dz}{z}$$

(ii) Mostrare che $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{1-\bar{z}w}\, \frac{dz}{z} = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z-w}\, dz$

(questo è abbastanza ovvio)

(iii) Mostrare che $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(z)}{z-w}\, dz = f(w)$.

(Si può fare con o senza residui).

(iv) Dai punti (i)-(iii) ricaviamo che

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1 - w \cdot e^{-i\theta}}\, d\theta = f(w).$$

Date f, g olomorfe in Ω, poniamo

$$\langle f, g \rangle_{H^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})}\, d\theta.$$

Dedurre da quanto visto sopra che

(*) $\langle f, K_w \rangle_{H^2} = f(w)$ per ogni f olomorfa in Ω e per ogni $w \in \Delta$,

se $K_w(z) = \frac{1}{1 - z\bar{w}}$.

(La formula (*) è la formula riproducente per lo spazio di Hilbert H^2).

(6) Sia $a \in \Delta$. Mostrare che $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{1-\bar{a}z}{z-a}\, dz = 1 - |a|^2$.

Se $|a| > 1$, quanto vale $\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{1-\bar{a}z}{z-a}\, dz$? [Risposta: 0]