One exercise

$$I = \int_{-\infty}^{+\infty} \frac{dx}{1+x^4} \qquad I_R = \int_{-R}^{R} \frac{dx}{1+x^4} \quad (R \geq 0)$$

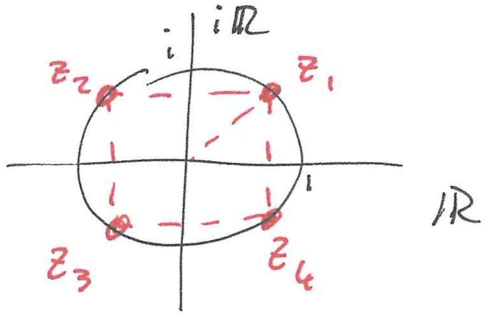
I will compute $\frac{1}{2\pi i}\int_\Gamma f(z)\,dz$ with $f(z) = \frac{1}{1+z^4}$ and Γ to be decided.

$z^4 + 1 = 0 \qquad z^4 = -1 \quad : \quad z = \rho e^{i\sigma}$:

$$\rho^4 e^{4i\sigma} = z^4 = -1 = e^{\pi i} \iff \begin{cases} \rho^4 = 1 & (\rho \geq 0) \\ 4\sigma = \pi + 2k\pi & (k \in \mathbb{Z}) \end{cases}$$

$$\begin{cases} \rho = 1 \\ \sigma = \frac{\pi}{4} + k\frac{\pi}{2} & k = 0, 1, 2, 3 \end{cases}$$

Let $z_k = e^{(\pi/4 + k\pi/2)i}$: $z_1 = \cos\pi/4 + i\sin\pi/4 = \frac{1}{\sqrt2} + \frac{i}{\sqrt2}$



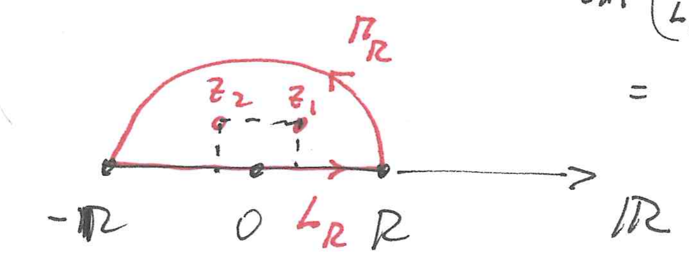
$z_2 = -\frac{1}{\sqrt2} + \frac{i}{\sqrt2}$

$z_3 = -\frac{1}{\sqrt2} - \frac{i}{\sqrt2}$

$z_4 = \frac{1}{\sqrt2} - \frac{i}{\sqrt2}$

$$\frac{1}{1+z^4} = f(z) = \frac{1}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

Parametrize $z = Re^{it}$; $0 \leq t \leq \pi$



$$\frac{1}{2\pi i}\left(\int_{L_R} + \int_{\Gamma_R}\right) f(z)\,dz =$$

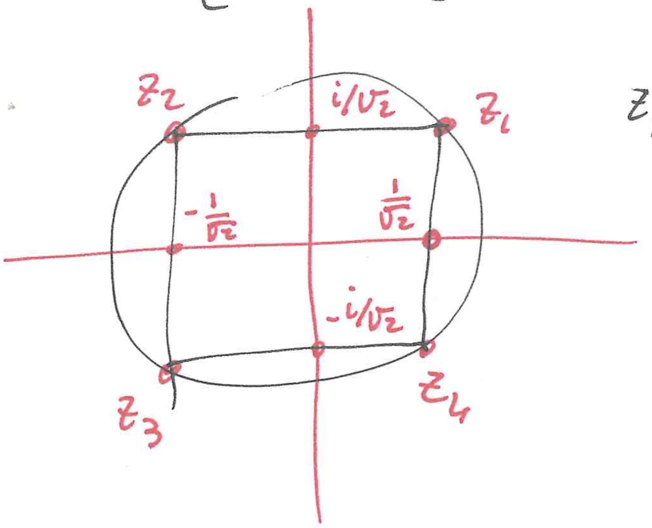
$$= \frac{1}{2\pi i}\left\{\int_{-R}^{R} \frac{dx}{1+x^4} + \int_0^\pi \frac{Rie^{it}\,dt}{1+R^4e^{4it}}\right\}$$

$$\frac{1}{2\pi i}\cdot I_R + \frac{iR}{2\pi i}\int_0^\pi \frac{e^{it}\,dt}{1+R^4e^{4it}} =$$

$$= \mathrm{Res}(f; z_1) + \mathrm{Res}(f; z_2) = \lim_{z\to z_1}(z-z_1)f(z) + \lim_{z\to z_2}(z-z_2)f(z)$$

$$= \frac{1}{(z_1-z_2)(z_1-z_3)(z_1-z_4)} + \frac{1}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$= \frac{1}{z_1 - z_2} \cdot \left[\frac{1}{(z_1 - z_3)(z_1 - z_4)} + \frac{1}{(z_2 - z_3)(z_2 - z_4)} \right] \overset{(*)}{=}$$



$$z_1 - z_2 = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$z_1 - z_3 = \frac{2}{\sqrt{2}} + \frac{2i}{\sqrt{2}} = \sqrt{2}(1+i)$$

$$z_1 - z_4 = \frac{2i}{\sqrt{2}} = \sqrt{2}\, i$$

$$z_2 - z_3 = \sqrt{2}\, i$$

$$z_2 - z_4 = 2\left(\frac{i}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \sqrt{2}(i-1)$$

$$(*) = \frac{1}{\sqrt{2}} \cdot \left[\frac{1}{\sqrt{2}(1+i)\ \sqrt{2}\, i} + \frac{1}{\sqrt{2}\, i\ \sqrt{2}(i-1)} \right]$$

$$= \frac{1}{\sqrt{2}^3} \cdot \frac{1}{i} \cdot \frac{(i-1)+(1+i)}{(1+i)(i-1)} = \frac{-2}{2\sqrt{2}\, i\ (i^2-1)} = + \frac{1}{2\sqrt{2}\, i}$$

Now: $\left| \int_0^{\pi} \frac{R i e^{it}\, dt}{1 + R^4 e^{4it}} \right| \leq \int_0^{\pi} \frac{R}{R^4 - 1}\, dt$ (if $R > 1$)

$$= \frac{\pi \cdot R}{R^4 - 1} \xrightarrow[R \to +\infty]{} 0$$

Thus: $I = \int_{-\infty}^{+\infty} f(x)\, dx = \lim_{R \to \infty} \left(\int_{L_R} + \int_{\Gamma_R} \right) f(z)\, dz$

$$= 2\pi i \cdot \frac{1}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

$$\boxed{I = \frac{\pi}{\sqrt{2}}}$$