Sobolev spaces. I

We want to have "weak derivatives" for cases where we can not integrate by parts. Functions with weak derivatives provide solutions to "weak differential equations".

Ascoli–Arzelà Thm. $\Omega \subseteq \mathbb{R}^n$ bounded open.

$K \subseteq C(\bar{\Omega})$ bounded and equicontinuous:

(i) $\|\varphi\|_\infty \leq \text{const} \quad \forall \varphi \in K$

(ii) $\forall \varepsilon > 0 \ \exists \delta = \delta(\varepsilon) > 0 \ \forall \varphi \in K \ \forall x, y \in \bar{\Omega}$:

$$|x - y| \leq \delta \Rightarrow |\varphi(x) - \varphi(y)| \leq \varepsilon$$

Then, $\bar{K}$ is compact in $C(\bar{\Omega})$.

i.e. if $\{\varphi_\nu\} \subseteq K$ then

$\exists \{\varphi_{\nu_k}\}$ sub.seq. of $\{\varphi_\nu\}$ s.t.

$$\varphi_{\nu_k} \xrightarrow[k \to \infty]{} \exists \varphi \in C(\bar{\Omega})$$

in $C(\bar{\Omega})$.

Memo. $C(\bar{\Omega})$ is a Banach space w.r.t. the norm

$$\|\varphi\|_\infty = \sup_{x \in \bar{\Omega}} |\varphi(x)|.$$

Convergence in $\|\cdot\|_\infty$ is uniform convergence.

Weak derivative.

Df. $\Omega \subseteq \mathbb{R}^n$ open and $u \in L^1_{loc}(\Omega)$. Let $v \in L^1_{loc}(\Omega)$. We say that v is the weak derivative of u w.r.t. x_j if

$$\int_\Omega v(x)\varphi(x)\,dx = -\int_\Omega u(x)\frac{\partial \varphi}{\partial x_j}(x)\,dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

We write $v = \dfrac{\partial u}{\partial x_j}$.

Examples.

(1) $u(x) = |x|$ has $u'(x) = \operatorname{sgn}(x)$ on $\mathbb{R}$.

(2) $u(x) = \operatorname{sgn}(x)$ has no weak derivative in $L^1_{loc}(\mathbb{R})$

(1) $\displaystyle\int_{\mathbb{R}} |x|\,\varphi'(x)\,dx = \left(\int_0^R + \int_{-R}^0\right)|x| \cdot \varphi'(x)\,dx$

$$= \left[x\varphi(x)\right]_0^R - \int_0^R \varphi(x)\,dx + \left[-x\varphi(x)\right]_{-R}^0 - \int_{-R}^0 (-\varphi(x))\,dx$$

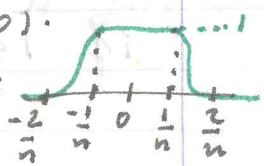
$$= -\int_{-R}^R \begin{Bmatrix} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{Bmatrix} \varphi(x)\,dx$$

$$= -\int_{\mathbb{R}} \operatorname{sign}(x)\,\varphi(x)\,dx.$$

Here R is s.t. $\varphi(x) = 0 \Leftarrow |x| \geq R$.

(2) $\displaystyle \int_{\mathbb{R}} v(x)\varphi(x)\,dx \overset{?}{=} -\int_{\mathbb{R}} \operatorname{sgn}(x)\,\varphi'(x)\,dx$

$$= -\int_0^\infty \varphi'\,dx + \int_{-\infty}^0 \varphi'\,dx = 2\varphi(0).$$

If such v existed, let $\varphi_n(x) =$ 

Then $2 = 2\varphi_n(0) = \int v\,\varphi_n\,dx \xrightarrow[n \to \infty]{} 0$ by monotone convergence Theorem.

Def. Let $\Omega \subseteq \mathbb{R}^n$ open and $1 \le p \le \infty$.

$W^{1,p}(\Omega) \ni v : \Omega \to \mathbb{R}$ if

(i) $v \in L^p(\Omega)$

(ii) $\exists\ \partial_j v \in L^p(\Omega)$ for $j = 1, \dots, n$

The corresponding norm is

$$\|v\|_{W^{1,p}} = \left(\|v\|^p_{L^p} + \|\nabla v\|^p_{L^p}\right)^{1/p} \text{ if } 1 \le p < \infty$$

$$\|v\|_{W^{1,\infty}} = \max\left(\|v\|_{L^\infty}, \|\nabla v\|_{L^\infty}\right) \text{ if } p = \infty$$

Def. $W^{1,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ w.r.t. $\|\cdot\|_{W^{1,p}}$ if $1 \le p < \infty$.

Theorem A $\Omega \subseteq \mathbb{R}^n$ open

(i) $W^{1,p}(\Omega)$ is a Banach space if $1 \le p \le \infty$

(ii) $W^{1,p}(\Omega)$ is separable if $1 \le p < \infty$

(iii) $W^{1,p}(\Omega)$ is reflexive if $1 < p < \infty$

(iv) $W^{1,2}(\Omega)$ is a Hilbert space w.r.t.

$$\langle v, w \rangle_{W^{1,2}} = \int_\Omega v w\, dx + \int_\Omega \nabla v \cdot \nabla w\, dx$$

(v) $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$ if $1 \le p < \infty$

The key to most of these results is regularization. Another important tool is localization.

Localization Lemma. $\Omega' \subseteq \Omega$ open and $v, w \in L^1_{loc}(\Omega)$, $\partial_j v = w$ in Ω (weakly)

$$\Longrightarrow \partial_j \left(v|_{\Omega'}\right) = w|_{\Omega'} \text{ in } \Omega' \text{ (weakly).}$$

Proof. Exercise on the def. of weak derivative.

Regularization Lemma. If $\Omega' \subset\subset \Omega$ and $\phi \in C^1_0(\mathbb{R}^n)$; $\int \phi\, dx = 1$; $\phi \ge 0$ and $\phi_\varepsilon(x) := \frac{1}{\varepsilon} \phi\left(\frac{x}{\varepsilon}\right)$ $(\varepsilon > 0)$, then

$$\forall v \in W^{1,p}(\Omega) \Longrightarrow \phi_\varepsilon * v \xrightarrow[\varepsilon \to 0]{} v \text{ in } W^{1,p}(\Omega')$$

Proof. We first show that

$$\boxed{\begin{aligned}\partial_{x_j}(\phi_\varepsilon * v) &= (\partial_{x_j} \phi_\varepsilon) * v \\ &= \phi_\varepsilon * (\partial_{x_j} v) \text{ in } W^{1,p}(\Omega').\end{aligned}}$$

Let $\psi \in C^1_0(\Omega')$ and $\varepsilon < \operatorname{dist}(\Omega, \Omega')$.

Then $\int_{\Omega'} \phi_\varepsilon * v(x) \cdot \partial_{x_j} \psi(x)\, dx = \iint v(x-y)\, \phi_\varepsilon(y)\, \partial_j \psi(x)\, dx\, dy$

$= \int \phi_\varepsilon(y) \cdot \int v(x-y)\, \partial_j \psi(x)\, dx \cdot dy$

$= -\int \phi_\varepsilon(y) \cdot \int \partial_j v(x-y)\, \psi(x)\, dx\, dy$

$= \dots = -\int (\phi_\varepsilon * \partial_j v)(x)\, \psi(x)\, dx$ ■

Then, $\| \phi_\varepsilon * u - u \|^p_{W^{1,p}(\Omega')} =$

$$= \| \phi_\varepsilon * u - u \|^p_{L^p} + \sum_j \| \partial_j(\phi_\varepsilon * u) - \partial_j u \|^p_{L^p}$$

$$= \| \phi_\varepsilon * u - u \|^p_{L^p} + \sum_j \| \phi_\varepsilon * \partial_j u - \partial_j u \|^p_{L^p}$$

$\underset{\varepsilon \to 0}{\longrightarrow} 0$ since $\| v * \phi_\varepsilon - v \|_{L^p} \underset{\varepsilon \to 0}{\longrightarrow} 0$ $\forall v \in L^p$. $\boxed{}$

The latter property belongs to L^p theory. Here is a proof.

Lemma. Let $\phi \in C(\mathbb{R}^n)$, $\phi > 0$, $\phi(x) \geq 0$ $\forall x$, $\phi(x) = 0$ for $|x| \geq 1$, $\int \phi(x)\,dx = 1$ and $\phi_\varepsilon(x) = \frac{1}{\varepsilon}\phi(x/\varepsilon)$.

Then

(i) $\forall \psi \in C_0(\mathbb{R}^n) \Rightarrow \psi * \phi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} \psi$ ~~in~~ uniformly on $\mathbb{R}^n$

(ii) $\forall v \in L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ $\Rightarrow v * \phi_\varepsilon \underset{\varepsilon \to 0}{\longrightarrow} v$ in L^p.

Pf. (i) Suppose $\psi \in C_0(\mathbb{R}^n)$, $\psi(x) = 0$ if $|x| \geq R$.

$$\psi * \phi_\varepsilon(x) - \psi(x) = \int_{\mathbb{R}^n} [\psi(x-y) - \psi(x)]\, \phi_\varepsilon(y)\,dy$$

because $\int \phi_\varepsilon(y)\,dy = \int \phi(y)\,dy = 1$

$$= \int_{|y| \leq \varepsilon} [\psi(x-y) - \psi(x)] \cdot \phi_\varepsilon(y)\,dy$$

For fixed $\eta > 0$ let $\varepsilon(\eta) > 0$: $0 < \varepsilon \leq \varepsilon(\eta)$

$\Rightarrow |\psi(x-y) - \psi(x)| \leq \eta$ $\quad \forall x \in \mathbb{R}^n$ $\forall y: |y| \leq \varepsilon$.

This is possible because ψ is uniformly continuous.

Then $0 < \varepsilon \leq \varepsilon(\eta) \Rightarrow |\psi * \phi_\varepsilon(x) - \psi(x)| \leq \eta$ $\quad \forall x \in \mathbb{R}^n$.

(ii) We first prove it for $v = \psi \in C_0(\mathbb{R}^n)$ as above. We have that

$$|\psi * \phi_\varepsilon(x) - \psi(x)| \leq \begin{cases} \eta & \text{if } \varepsilon \leq \varepsilon(\eta) \\ 0 & \text{if } |x| \geq R + \varepsilon(\eta). \end{cases}$$

$$\Rightarrow \| \psi * \phi_\varepsilon - \psi \|_{L^p} \leq \text{Volume } B(0, R + \varepsilon(\eta)) \cdot \eta^p \underset{\eta \to 0}{\longrightarrow} 0.$$

For $v \in L^p$ and $\psi \in C_0(\mathbb{R}^n)$ we have

$$\| v * \phi_\varepsilon - v \|_{L^p} = \| v * \phi_\varepsilon - \psi * \phi_\varepsilon + \psi * \phi_\varepsilon - \psi + \psi - v \|_{L^p}$$

$$\leq \| (v - \psi) * \phi_\varepsilon \|_{L^p} + \| \psi * \phi_\varepsilon - \psi \|_{L^p} + \| \psi - v \|_{L^p}.$$

Fix $\eta > 0$ and choose $\psi \in C_0(\mathbb{R}^n)$ s.t.

$\| \psi - v \|_{L^p} < \eta$ (you can because $C_0(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$)

Make use of the easy Young's inequality

(Y) $\| f * g \|_{L^p(\mathbb{R}^n)} \leq \| f \|_{L^p(\mathbb{R}^n)} \cdot \| g \|_{L^1(\mathbb{R}^n)}$

to obtain

$\| v * \phi_\varepsilon - v \|_{L^p} \leq$

$\leq \| v - \psi \|_{L^p} \cdot \| \phi_\varepsilon \|_{L^1} + \| \phi_\varepsilon * \psi - \psi \|_{L^p} + \| v - \psi \|_{L^p}$

$\leq 2\eta + \| \phi_\varepsilon * \psi - \psi \|_{L^p}$.

Since $\psi \in C_0(\mathbb{R}^n)$, for $0 < \varepsilon \leq \varepsilon(\eta)$

we have $\| \phi_\varepsilon * \psi - \psi \|_{L^p} \leq \eta$

and we are done. ∎

Proof of the easy inequality (Y)

$\| f * g \|_{L^p} = \left\{ \int \left| \int f(x-y) g(y) \, dy \right|^p dx \right\}^{1/p}$

$\leq \int \left(\int |f(x-y)|^p dx \right)^{1/p} \cdot |g(y)| \, dy$

by Minkowsky's inequality

$= \int \| f \|_{L^p} \, |g(y)| \, dy$ by translation invariance of dx

$= \| f \|_{L^p} \cdot \| g \|_{L^1}$. ∎

Memo: some basic inequalities in L^p theory.

Hölder's inequality.

$1 \leq p, p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$

$\Rightarrow \left| \int f \cdot g \, dx \right| \leq \| f \|_{L^p} \cdot \| g \|_{L^{p'}}$

Minkovsky's inequality. $1 \leq p < \infty$.

If $f \geq 0$ is measurable on $\mathbb{R}^n \times \mathbb{R}^m$,

$\left\{ \int \left| \int f(x,y) \, d\mu(y) \right|^p d\lambda(x) \right\}^{1/p} \leq$

$\leq \int \left\{ \int f(x,y)^p \, d\lambda(x) \right\}^{1/p} d\mu(y)$

for Borel measures λ on $\mathbb{R}^n$, μ on $\mathbb{R}^m$.

The statement is "obvious":

$\left\{ \int \left| \sum f_j(x) \mu_j \right|^p d\lambda(x) \right\}^{1/p} =$

$= \left\| \sum f_j \mu_j \right\|_{L^p} \leq \sum |\mu_j| \cdot \| f_j \|_{L^p}$

$= \sum |\mu_j| \cdot \left(\int |f_j(x)|^p \, d\lambda(x) \right)^{1/p}$

if $f_j \in L^p$ for $j = 1, \ldots, r$

and $\mu_j \in \mathbb{C}$ $j = 1, \ldots, r$,

since $\| \cdot \|_p$ is a norm.

Minkovsky's inequality is the limiting (integral) version of this.