Direct method: the Dirichlet integral.

Theorem. Let $\Omega \subseteq \mathbb{R}^n$ be bounded, open with Lipschitz (or even C^1) boundary. Let $v_0 \in W^{1,2}(\Omega)$ and let

$$m := \inf\left\{ I(v) = \frac{1}{2}\int_\Omega |\nabla v(x)|^2\,dx \quad \text{as} \quad v - v_0 \in W_0^{1,2}(\Omega)\right\}.$$

There is exactly one $\bar{v} \in v_0 + W_0^{1,2}(\Omega)$ such that $I(\bar{v}) = m$.

Moreover $\bar{v}$ satisfies the weak form of Laplace equation

$$\text{(weak)} \quad \int_\Omega \nabla\bar{v}\cdot\nabla\varphi\,dx = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega).$$

Conversely if $\bar{v} \in v_0 + W_0^{1,2}$ satisfies (weak) then $I(\bar{v}) = m$.

Proof. Existence: the main idea is reproducing the proof of Weierstrass Thm.
Uniqueness: here we use convexity.

Existence.
Step 1. We find a sequence $\{v_\nu\}$ in $v_0 + W_0^{1,2}(\Omega)$: $v_\nu \xrightarrow[\nu\to\infty]{} \bar{v}$ (weakly).

Observe that $0 \le m \le I(v_0 + 0) < \infty$ since $v_0 \in W^{1,2}(\Omega)$.
Let $\{v_\nu \in v_0 + W_0^{1,2}(\Omega)\}_{\nu\in\mathbb{N}}$ s.t.

$$I(v_\nu) \xrightarrow[\nu\to\infty]{} m$$

$$\|v_\nu\|_{W^{1,2}} \le \|v_\nu - v_0\|_{W_0^{1,2}} + \|v_0\|_{W^{1,2}}$$

since $v_\nu - v_0 \in W_0^{1,2}$

$$\le \gamma_1\left(\int_\Omega |\nabla(v_\nu - v_0)|^2 dx\right)^{1/2} + \|v_0\|_{W^{1,2}}$$

by Poincaré inequality

$$\le \gamma_2\left(\int_\Omega |\nabla v_\nu|^2 dx\right)^{1/2} + \gamma_3\|v_0\|_{W^{1,2}}$$

$$= \gamma_2\sqrt{2I(v_\nu)} + \gamma_3\|v_0\|_{W^{1,2}} \le \gamma_4$$

From Banach–Alaoglu we can find a subsequence $\{\nu_k\}$ s.t.

$$v_{\nu_k} \xrightarrow[k\to\infty]{} \bar{v} \in W^{1,2}(\Omega); \quad \|\bar{v}\|_{W^{1,2}(\Omega)} \le \gamma_4.$$

Relabel $\nu_k \leftrightarrow \nu$.

Step 2. We show that

$$u_\nu \underset{\nu\to\infty}{\rightharpoonup} \bar u \text{ in } W^{1,2} \Rightarrow \underline{\lim}_{\nu\to\infty} I(u_\nu) \geq I(\bar u).$$

Start with

$$|\nabla u_\nu|^2 = |\nabla \bar u|^2 + 2\,\nabla \bar u \cdot \nabla(u_\nu - \bar u) + |\nabla(u_\nu - \bar u)|^2$$
$$\geq |\nabla \bar u|^2 + 2\,\nabla \bar u \cdot \nabla(u_\nu - \bar u),$$

~~from which~~ implying

$$I(u_\nu) \geq I(\bar u) + \int_\Omega \nabla \bar u \cdot \nabla(u_\nu - \bar u)\,dx$$

Now, $u_\nu - \bar u \underset{\nu\to\infty}{\rightharpoonup} 0$ in $W^{1,2}$ means that $\forall \varphi \in W^{1,2}$ we have

$$0 = \lim_{\nu\to\infty} \langle u_\nu - \bar u, \varphi\rangle_{W^{1,2}} =$$
$$= \lim_{\nu\to\infty} \left\{ \int_\Omega \nabla(u_\nu - \bar u)\cdot\nabla\varphi\,dx + \int_\Omega (u_\nu - \bar u)\,\varphi\,dx \right\}$$

~~Let now $\varphi = \bar u$:~~

$$\cancel{0 = \lim_{\nu\to\infty} \left\{ \int_\Omega \nabla(u_\nu - \bar u)\cdot\nabla \bar u + \int_\Omega (u_\nu - \bar u)\,\bar u\,dx \right\}}$$

Lemma. $\lim_{\nu\to\infty} \langle u_\nu - \bar u, \varphi\rangle_{W^{1,2}} = 0$

$\Leftrightarrow$ (i) $\lim_{\nu\to\infty} \langle u_\nu - \bar u, \varphi\rangle_{L^2} = 0$

and (ii) $\lim_{\nu\to\infty} \langle \nabla(u_\nu - \bar u), \nabla\varphi\rangle_{L^2} = 0$

Then $\int_\Omega \nabla(u_\nu - \bar u)\nabla \bar u\,dx \underset{\nu\to\infty}{\longrightarrow} 0$

hence

$$\underline{\lim}_{\nu} I(u_\nu) \geq I(\bar u) + \lim_\nu \int_\Omega \nabla \bar u \cdot \nabla(u_\nu - \bar u)$$
$$= I(\bar u) \quad \text{as wished.}$$

By Steps 1 and 2 we have

$$m \leq I(\bar u) \leq \underline{\lim}\, I(u_\nu) = m,$$

hence $\bar u$ is a minimizer.

Uniqueness. Suppose $\bar u, \bar v \in u_0 + W_0^{1,2}(\Omega)$ with $I(\bar u) = I(\bar v) = m$. Hence,

$\frac{\bar u + \bar v}{2} \in u_0 + W_0^{1,2}(\Omega)$ and

$$m \leq I\left(\frac{\bar u + \bar v}{2}\right) = \frac{1}{2}\int_\Omega \left|\frac{\nabla \bar u + \nabla \bar v}{2}\right|^2 dx$$
$$\leq \frac{1}{2}\int_\Omega \frac{|\nabla \bar u|^2 + |\nabla \bar v|^2}{2}\,dx \quad \text{since } \xi \mapsto |\xi|^2 \text{ is convex}$$
$$= \frac{I(\bar u) + I(\bar v)}{2} = m$$

hence $\frac{\bar u + \bar v}{2}$ is minimizing, too. So,

$$0 = \int_\Omega \frac{|\nabla \bar u|^2 + |\nabla \bar v|^2}{2} - \left|\frac{\nabla \bar u + \nabla \bar v}{2}\right|^2 dx$$

but the integrand is ≥ 0 by convexity

$$\Rightarrow \frac{|\nabla \bar u|^2 + |\nabla \bar v|^2}{2} - \left|\frac{\nabla \bar u + \nabla \bar v}{2}\right|^2 = 0 \quad \text{a.e.}$$

$\Rightarrow$ (by strict convexity of $\xi \mapsto |\xi|^2$)

$\nabla \bar u = \nabla \bar v$ a.e. $\Rightarrow$ $\bar u - \bar v = 0$ a.e.

We have used two facts.

Lemma. $\nabla v = 0$ a.e. and $v \in W^{1,2}(\Omega)$
$\Rightarrow \exists c \in \mathbb{R} : v = c$ a.e.

Lemma. $v = c$ a.e. and $v \in W_0^{1,2}(\Omega)$
$\Rightarrow c = 0$.

Euler-Lagrange equation.

Supp. $\bar{v}$ is a minimizer and $\varphi \in W_0^{1,2}(\Omega)$. Let $\varepsilon \in \mathbb{R}$. Then

$$I(\bar{v}) \leq I(\bar{v} + \varepsilon\varphi) = \frac{1}{2}\int_\Omega |\nabla \bar{v} + \varepsilon \nabla\varphi|^2 dx$$

$$= I(\bar{v}) + \int_\Omega \nabla\bar{v} \cdot \nabla\varphi \, dx \cdot \varepsilon + I(\varphi)\varepsilon^2$$

i.e. $\quad 0 \leq \varepsilon\left(\int_\Omega \nabla\bar{v} \cdot \nabla\varphi \, dx + \varepsilon I(\varphi)\right)$

$$\Rightarrow \underset{\text{if } \varepsilon > 0}{0 \leq} \int_\Omega \nabla\bar{v}\cdot\nabla\varphi \, dx + \varepsilon I(\varphi) \underset{\text{if } \varepsilon < 0}{\leq 0}$$

$$\Rightarrow \boxed{\int_\Omega \nabla\bar{v}\cdot\nabla\varphi \, dx = 0} \quad \forall \varphi \in W_0^{1,2}(\Omega)$$

Obs. that if we knew that $\bar{v} \in C^2(\bar{\Omega})$, we could use div. Thm. to show

$$0 = \int_\Omega \nabla\bar{v}\cdot\nabla\varphi \, dx = \int_{\partial\Omega} \varphi \, \partial_\nu \bar{v} \, d\sigma - \int_\Omega \Delta\bar{v}\cdot\varphi \, dx$$

$$= -\int_\Omega \Delta\bar{v}\cdot\varphi \, dx \quad \forall \varphi \in C_0^\infty(\Omega)$$

$\Rightarrow \Delta\bar{v} = 0$.

For this reason

$$\int_\Omega \nabla\bar{v}\cdot\nabla\varphi \, dx = 0 \quad \forall\varphi \in W_0^{1,2}(\Omega)$$

is called the weak form of $\Delta\bar{v} = 0$.

Converse statement.

Supp. $\bar{v} \in v_0 + W_0^{1,2}(\Omega)$ satisfies weak, let $v \in v_0 + W_0^{1,2}(\Omega)$ and $\varphi = v - \bar{v} \in W_0^{1,2}(\Omega)$. Then

$$I(v) = I(\bar{v} + \varphi) = I(\bar{v}) + \int_\Omega \nabla\bar{v}\cdot\nabla\varphi \, dx + I(\varphi)$$

$$= I(\bar{v}) + I(\varphi) \qquad \text{by (weak)}$$

$$\geq I(\bar{v})$$

i.e. $\bar{v}$ is a minimizer. ∎

Exercise. Let $I(v) = \frac{1}{p}\int_\Omega |\nabla v|^p dx$; $1 < p < \infty$.

(i) Show that $\xi \mapsto |\xi|^p$ is convex on $\mathbb{R}^n$; hence that $I(tv + (1-t)\tilde{v}) \leq t \cdot I(v) + (1-t) I(\tilde{v})$
$\forall v, \tilde{v} \in W^{1,p}(\Omega) \quad \forall t \in [0,1]$

(ii) Find weak and strong form of the Euler-Lagrange equation for I (it's called the p-Laplace eq.).

(iii) Reproduce the proof of the Thm. just proved and provide a more general statement.