ESERCIZIO sul FLUSSO

$$\Omega = \left\{ (x,y,z) : \frac{x^2}{4} + y^2 \le 1 \; ; \; \frac{x^2}{4} + y^2 + z^2 \ge 1 ; \; z \le 4 \right\}$$

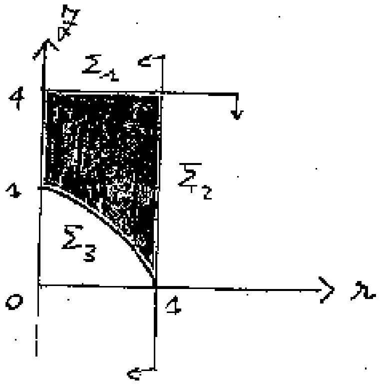
1. Parametrizzare $\partial\Omega$ compatibilmente con la normale esterna a Ω.
2. Sia $\Sigma = \partial\Omega \setminus \left\{ (x,y,4) : \frac{x^2}{4} + y^2 < 1 \right\}$

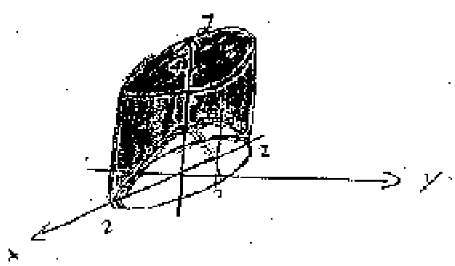
   calcolare il flusso $\iint_{(\Sigma,\nu)} F \cdot d\sigma$ con $F(x,y,z) = (x, zy, 0)$

Uso coordinate cilindriche in versione ellittica

$$\begin{cases} \frac{x}{2} = r\cos\vartheta \\ y = r\,\mathrm{sen}\,\vartheta \\ z = z \end{cases} \qquad \frac{x^2}{4} + y^2 = r^2$$

$$r \ge 0 \;,\; r^2 \le 1 \;,\; r^2 + z^2 \ge 1 \;,\; z \le 4$$





Parametrizzazioni

$\Sigma_1 = \{(x,y,4) : \frac{x^2}{4} + y^2 \le 1\} \subseteq \mathbb{R}^3$

$\Omega_1 : \{(x,y) : \frac{x^2}{4} + y^2 \le 1\} \subseteq \mathbb{R}^2 \xrightarrow{\bar{\Phi}_1} \mathbb{R}^3$

$\bar{\Phi}_1(x,y) = (x,y,4)$

$J\bar{\Phi}_1(x,y)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \partial_x \bar{\Phi}_1(x,y) \times \partial_y \bar{\Phi}_1(x,y) =$

$\begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = k = (0,0,1)$

punta verso l'esterno $\Rightarrow \bar{\Phi}_1$ è compatibile

$\Sigma_2 = \{(x,y,z) : \frac{x^2}{4} + y^2 = 1 \; ; \; 0 \le z \le 4\}$

$\bar{\Phi}_2(\vartheta, z) = (2\cos\vartheta, \operatorname{sen}\vartheta, z)$

$\bar{\Phi}_2 : [0,2\pi] \times [0,4] \longrightarrow \mathbb{R}^3$

$\Omega_2 : \{(\vartheta, z) : 0 \le \vartheta \le 2\pi, \; 0 \le z \le 4\} \longrightarrow \mathbb{R}^3$

$\cap$
$\mathbb{R}^2$

$\bar{\Phi}_2(\Omega_2) = \Sigma_2$

$J\bar{\Phi}_2(\vartheta, z)^T = \begin{pmatrix} -2\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$

$\partial_\vartheta \bar{\Phi}_2(\vartheta,z) \times \partial_z \bar{\Phi}_2(\vartheta,z) = \begin{vmatrix} i & j & k \\ -2\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \circledast$

Per $\vartheta = 0$

$\circledast = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \end{vmatrix} = i = (1,0,0)$

$$\Phi_2(0,z) = (2,0,z) \underset{\text{per } z=0}{=} (2,0,0)$$
$$y=0$$

punta fuori $\Rightarrow \Phi_2(\vartheta,z)$ è compatibile

$$\Sigma_3 = \left\{(x,y,z) : \frac{x^2}{4}+y^2+z^2=1,\ z\geq 0\right\} \qquad z^2 = 1-r^2 \rightarrow z = \pm\sqrt{1-r^2}$$

$$\Phi_3(\vartheta,r) = (r\cos\vartheta,\ r\sin\vartheta,\ z=\sqrt{1-r^2})$$

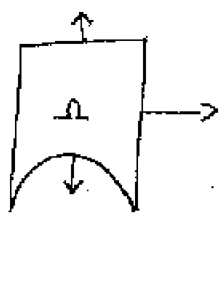
$$\Phi_3(\vartheta,r) : \{(\vartheta,r) : 0\leq\vartheta\leq 2\pi;\ 0\leq r\leq 1\} =$$
$$= [0,2\pi]\times[0,1] = \Omega_3 \longrightarrow \mathbb{R}^3$$

$$J\Phi_3(\vartheta,r)^T = \begin{pmatrix} -r\sin\vartheta & r\cos\vartheta & 0 \\ \cos\vartheta & \sin\vartheta & \frac{-r}{\sqrt{1-r^2}} \end{pmatrix} \Rightarrow$$

$$\partial_\vartheta\Phi_3(\vartheta,r)\times\partial_r\Phi_3(\vartheta,r) \underset{\vartheta=0}{=} \begin{vmatrix} i & j & k \\ 0 & r & 0 \\ 1 & 0 & \frac{-r}{\sqrt{1-r^2}} \end{vmatrix} =$$

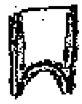
$$= i\left(\frac{-r^2}{\sqrt{1-r^2}}\right) + k(-r)$$

La terza componente di $\nu(\Phi_3(\vartheta,r))$ è negativa $\Rightarrow$ punta verso il basso, cioè l'esterno di $\Sigma_3 \Rightarrow$ è compatibile



normale uscente

(2) $\Sigma = \partial\Omega \setminus$ "coperchio"



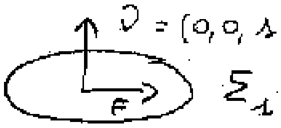
$\Sigma = \Sigma_2 \cup \Sigma_3$

$$\iint_{(\Sigma,\nu)} F\cdot\nu \, d\sigma = \iint_{(\partial\Omega,\nu)} F\cdot\nu \, d\sigma - \iint_{(\Sigma_1,\nu)} F\cdot\nu \, d\sigma =$$

$$= \iiint_{\Omega} (\operatorname{div} F) \, dx\,dy\,dz - \iint_{(\Sigma_1,\nu)} F\cdot\nu\, d\sigma$$

$\| \operatorname{div} F = 3$

$$3 \iiint_{\Omega} 1 \, dx\,dy\,dz - \iint_{(\Sigma_1,\nu)} F\cdot\nu \, d\sigma \quad (*)$$



$\Rightarrow \quad F\cdot\nu = (x, 2y, 0)(0,0,1) = 0$

$F = (x, 2y, 0)$

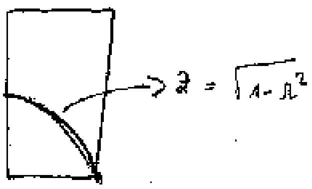
$$\overset{(*)}{=} 3 \iiint_{\Omega} 1 \, dx\,dy\,dz - 0$$

Rimane da calcolare solo il volume di Ω e moltiplicarlo per 3

$\Omega \ni (x, y, z) = (2r\cos\vartheta, r\sin\vartheta, z) \Leftrightarrow$

$0 \le \vartheta \le 2\pi \quad e \quad 0 \le r \le 1$

$e \quad \sqrt{1-r^2} \le z \le 4$



$$\mathrm{Vol}\,\Omega = \iiint_{\Omega} 1\,dx\,dy\,dz = \iiint 2r\;dr\,d\vartheta\,dz =$$ (5)

$A = \{(r,\vartheta,z) : 0 \le \vartheta \le 2\pi,\ 0 \le r \le 1,\ \sqrt{1-r^2} \le z \le 4\}$

$$\det J(\Phi(r,\vartheta)) = \begin{vmatrix} 2\cos\vartheta & -2r\sin\vartheta \\ \operatorname{sen}\vartheta & r\cos\vartheta \end{vmatrix} =$$

$$= 2r\cos^2\vartheta + 2r\sin^2\vartheta = 2r(\cos^2\vartheta + \sin^2\vartheta) = 2r$$

$$= \int_0^{2\pi} d\vartheta \int_0^1 2r\,dr \int_{\sqrt{1-r^2}}^{4} dz =$$

$$= 2\pi \cdot \int_0^1 2r\left(4 - \sqrt{1-r^2}\right) dr =$$

$$= 2\pi \int_0^1 8r - 2r\sqrt{1-r^2}\,dr = 2\pi\left\{\frac{8}{2}\left[r^2\right]_0^1 - \int_0^1 2r\sqrt{1-r^2}\,dr\right\} =$$

$$= 2\pi\left\{4\left[1-0\right] + \frac{2}{3}\left[(1-r^2)^{\frac{3}{2}}\right]_0^1\right\} = 2\pi\left\{4 + \frac{2}{3}\left[0-1\right]\right\} =$$

$$= 2\pi\left\{4 + \frac{2}{3}(-1)\right\} = 2\pi\left\{\frac{12-2}{3}\right\} = 2\pi \cdot \frac{10}{3}$$

$$\Rightarrow 3\iiint_{\Omega} 1\,dx\,dy\,dz = \cancel{3} \cdot \frac{20\pi}{\cancel{3}} = 20\pi$$

Altro modo per calcolare il flusso

$$\iint_{(\Sigma_2,\nu)} F\cdot\nu\,d\sigma = \int_0^{2\pi} d\vartheta \int_0^4 dz\ F(2\cos\vartheta, \sin\vartheta, z)\,(\partial_\vartheta\Phi_2 \times \partial_z\Phi_2)(\vartheta,z)$$

$$= \int_0^{2\pi} d\vartheta \int_0^4 dz\,(2\cos\vartheta, 2\sin\vartheta, 0)\cdot(\cos\vartheta, 2\sin\vartheta, 0) =$$

$$= \int_0^{2\pi} d\vartheta \int_0^4 2\cos^2\vartheta + 4\operatorname{sen}^2\vartheta\,dz = 4\int_0^{2\pi} 2\cos^2\vartheta + 4\operatorname{sen}^2\vartheta\,d\vartheta =$$

$$= 4\cdot(2\pi + 4\pi) = 24\pi$$

$$\Phi_3(\vartheta, r) = (2r\cos\vartheta, r\operatorname{sen}\vartheta, \sqrt{1-r^2}) \to$$

$$\partial_\vartheta \Phi_3(\vartheta, r) \times \partial_r \Phi_3(\vartheta, 3) = \begin{vmatrix} i & j & k \\ -2r\cos\vartheta & r\cos\vartheta & 0 \\ 2\cos\vartheta & \operatorname{sen}\vartheta & \frac{-r}{\sqrt{1-r^2}} \end{vmatrix} =$$

$$= \left(-\frac{r^2\cos\vartheta}{\sqrt{1-r^2}} , \frac{-2r^2\sin\vartheta}{\sqrt{1-r^2}} , -2r \right) = ⊛$$

$$\iint_{(\Sigma_3, \nu)} F \cdot \nu \, d\sigma = \int_0^{2\pi} d\vartheta \int_0^1 dr \, (2r\cos\vartheta, 2r\sin\vartheta, 0) \cdot ⊛ =$$

$$= \int_0^{2\pi} d\vartheta \int_0^1 dr \left(\frac{-2r^3\cos^2\vartheta}{\sqrt{1-r^2}} - \frac{4r^3\sin^2\vartheta}{\sqrt{1-r^2}} \right) =$$

$$= -\int_0^{2\pi} d\vartheta \int_0^1 dr \, \frac{2r^3\cos^2\vartheta}{\sqrt{1-r^2}} + \frac{4r^3\sin^2\vartheta}{\sqrt{1-r^2}} =$$

$$= -\left\{ \int_0^{2\pi} d\vartheta \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} (\cos^2\vartheta + 2\sin^2\vartheta) \, dr \right\} =$$

$$= -\left\{ \int_0^{2\pi} \cos^2\vartheta + \sin^2\vartheta + \sin^2\vartheta \, d\vartheta \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} dr \right\} =$$

$$= -\left\{ \left[\int_0^{2\pi} 1 \, d\vartheta + \int_0^{2\pi} \frac{1-\cos 2\vartheta}{2} d\vartheta \right] \cdot \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} dr \right\} =$$

$$= -\left\{ \left[2\pi + \frac{1}{\cancel{2}} \cdot \cancel{2}\pi + 0 \right] \cdot \int_0^1 \frac{2r^3}{\sqrt{1-r^2}} dr \right\} =$$

$$= -6\pi \int_0^1 \frac{r^3}{\sqrt{1-r^2}} dr = ⊛$$

Pongo $r^2 = t \to dt = 2r\,dr$

$$⊛ = -6\pi \int_0^1 \frac{t}{\sqrt{1-t}} dt = -3\pi \int_0^1 \frac{t-1+1}{\sqrt{1-t}} dt =$$

$= -3\pi \int_0^1 \frac{1}{\sqrt{1-t}}\, dt + 3\pi \int_0^1 \sqrt{1-t} =$ ②

$= -3\pi \left[-2 (1-t)^{\frac{1}{2}} \right]_0^1 + 3\pi \left(\frac{2}{3} (1-t)^{3/2} \right)_0^1 = -3\pi \left(2 - \frac{2}{3} \right) = \frac{4}{3} (-3\pi) =$

$= -4\pi$

Quindi

$\iint_{(\Sigma_2, \nu)} F \cdot \nu \cdot d\sigma + \iint_{(\Sigma_3, \nu)} F \cdot \nu \cdot d\sigma =$

$= 20\pi - 4\pi = 20 \pi$ c.v.d.